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Chapitre 1

General introduction

Quantum field theory is the unfinished coronation of quantum mechanics and the laws of relativity. In spite of the exploit of its experimental predictions, it remains full of divergences which one could not eliminate except by methods of regularization mathematics and physical renormalization. Consequently, this major concern and this disturbing trend were at the origin of the emergence of the theory of quantum deformation modeled by deformed algebras where the parameters of the deformation are considered as being cut-off of the theory.

During the last years, the deformed algebra plays an increasingly important role in various fields of physics and particularly in quantum field theory and it has become a very interesting perspective topic to physicists. In this regard, many various forms of deformed algebra were introduced and are used as model in several problems. Let us quote some : the description of the low energy excitations of graphene and the Fermi velocity , is based on a deformation of the Heisenberg algebra which makes the commutator of momenta proportional to the pseudo-spin[1]. The dynamics of systems with variable masses in semiconductor heterostructures are formulated by deformed quadratic algebra [2], the thermostatistics of q-deformed bosons and fermions [3], the q-deformed quark fields [4], the motion of a ^{3}He impurity atom in the Bose liquid [5]and an atom placed in a gravitational field [6].For this purpose, the relativistic and non-relativistic quantum mechanics with and without spin have developed significantly in different contexts notably in the framework of deformed algebras and many works are examined.

Historically, the first deformation known in the literature is non-commutative geometry proposed by Connes with a widely applications in different contexts. The conception of this noncommutativity is due in the first place goes back to Heisenberg and Snyder as a solution in order to absorb infinite quantities in field theories before the renormalization formalism [7]. Since its establishment, its has continued to evolve and meet the mathematical requirements of various situations effectively, as it has invaded certain domains of physics. In addition, the noncommutativity between coordinates appeared in string theory in relation with D-branes and the quantum Hall effect. Although the noncommutativity between coordinates produces features including the breaking of Lorentz invariance, UV/IR mixing phenomenon, the violation of unitarity and causality, there are considerable motivations to study in this direction. Since the development of string theory [8], several authors solved many problems with different methods in the framework of noncommutative space. For example, central potential [9, 10]. The Landau problem with a harmonic oscillator potential on the noncommutative plane and two-sphere are studied in [11] , and the conditions for the equivalence of the noncommutative quantum mechanics and the Landau problem are given in [12],Dirac and Klein–Gordon oscillators [13, 14, 15], Feshbach–Villars equation (spin 0) in interaction with a scalar potential [16] , magnetic field[17], the Landau problem [18] and hydrogen atom [19, 20].

Another important deformation so-called the generalized uncertainty principle (GUP) $[[21]-[25]]$can be obtained from a modified Heisenberg algebra (HUP), introduced in order to take into account the effects of the gravitational field , when we incorporate the gravity theory with quantum mechanic and it is characterized by the existence of a minimal length scale in the order of the Planck length . This deformation has been used in different problems as : black hole physics [26, 27],Bosonic oscillator in the presence of minimal length [28], A generalized Bosonic oscillator in the presence of a minimal length[29],Klein-Gordon Oscillator[30],the spinning particle subjected to the action of combined vector and scalar potentials[31],waves equations with different potentials [32],Scalar electrodynamics [33],the length scale of the material(graphen)[34],Dirac equation [35].

Besides of this deformed theories previously mentioned, its also exists a deformation linked to the topology of the physical space, in which the modified uncertainty principle associated called the extended uncertainty principle (EUP) [[36] $-$ [41]].In addition, in these research works, Mignemi showed that in a (Anti) de Sitter background the Heisenberg uncertainty principle modified by adding corrections proportional to the cosmological constant $\wedge = -3\lambda^2$, where $\lambda^2 \prec 0$ for de Sitter space-time, and $\lambda^2 \succ 0$ for anti-de Sitter space-time. The appearance of this idea (EUP) has drawn great attention and many papers have been published, we find :the effects of IR gravity on quantum mechanics[42],particles with position-dependent mass[43],the Ramsauer-Townsend effect in q-deformed quantum mechanics [44],the DKP oscillator with a linear interaction in the cosmic string space-time[45],the thermodynamic properties of the relativistic harmonic oscillators are investigated [46] , Bosonic oscillator under a uniform magnetic field with Snyder-de Sitter algebra[47],the corrections to Hawking temperature and Bekenstein entropy of a black hole for Rindler and cosmological horizons [48],Bosonic Oscillator on the de Sitter and the Anti-de Sitter Spaces[49] ,signals of the weak and strong deflection gravitational lensings are studied [50], the quantum gravity effects in the vicinity of a black hole[51] ,and the Klein–Gordon oscillator in an uniform magnetic field [52].

Finally and most recently, it is remarkable that, the presence of other forms of the deformation such as : the new type of EUP cases [53, 54, 55]and the "Doubly-Special-Relativity" (DSR) theories[[56] $-$ [67]]. We note that the new EUP has been introduced by the action of the translation operator in a space with a diagonal metric for the purpose of describing the motion of a quantum particle in curved space . For the DSR theory , it was proposed by J. Magueijo and L. Smolin [60, 61] and it was characterized by two observer independent large-velovity scale c, and large-momentum scale κ.

Similarly in this direction, the path integral formalism has particular interest and undergone notable development in various domains of physics with different topologies modeled by deformed algebras. At this level, the introduction of deformed algebras would probably be necessary . Because it makes naturally the relative scattering amplitudes to be ultra-violet regularized and it also gives some information on the regularity and the renorrmalization of statistical partition function where the parameters of deformation being cutoff of the theory. Some problems have found their solutions via Feynman path integral approach with only one deformation parameter . Consequently, in this regard, a significant number of papers have been published. Citing for instance, the spinning particle subjected to the action of combined vector and scalar potentials [31, 68] and the Dirac oscillator [69, 70, 71] are treated via the path integral approach with deformed GUP . In noncommutative space, the Klein-Gordon and Dirac oscillators [72], and the harmonic oscillator related to energy-dependent potential [73]. And others important similar references using path approach as : , the D-dimensional harmonic oscillator in [74], the Klein Gordon particle [75],the construct the kernel for a free particle by [76], the one dimension propagator for Dirac oscillator [69],the harmonic oscillator and the radial hydrogen atom propagators related to energy-dependent potentials are analyzed [77], the one dimension relativistic spinning particle with vector and scalar linear potentials in [31], the Klein-Gordon equation with the energy dependent linear and Coulomb potentials is treated in [78] the two dimensions relativistic Dirac oscillator [70], the one dimension harmonic oscillator by [79] and the Coulomb potential [80] . However, despite its successful results, Feynman approach still needs to be sharpened as a quantification tool since in the case of deformation or that of constraints we do not know a priory how to discretize the action without choosing the discretization procedure.

For pedagogical reasons, the main objectives of this work are the treatment some fundamental problems of relativistic and non relativistic quantum mechanics by two method of quantification : by a direct method , i.e. resolution of equations, and via the formalism of supersymmetric path integrals within the framework of deformed algebras. In order, to know the influence of this deformed theory in the physics result obtained as for example the phenomenon of confinement on the one hand. In other hand, we believe that Feynman's approach is not yet finalized despite its successes. It still requires a deep development, for example the case of systems with spin and their classical descriptions via the trajectories, the theory of constraints and the principle of discretization and the ambiguities encountered in the computation of quantum fluctuations within the framework of deformed algebras. Therefore, our attempt through this work, is then to analyze the example of the Dirac oscillator for spin $\frac{1}{2}$ in the context of the EUP.

In this thesis, we have organized our work in two essential parts : -The first part is composed of three chapters :

- In chapter 1, we give an explicit calculations of The Klein-Gordon equation (K-G) in the context of in the context of new type of the extended uncertainty principle and in the presence of certain interactions : K-G particle confined in a one dimensional box, in the scalar particle with linear vector and scalar potentials , in the Coulomb-type vector and scalar potentials. In all cases, we succeeded to determine the energy spectrum and the associated wave functions.
- In chapter 2, we have studied the one-dimensional K-G and Dirac oscillators problems in the presence of a uniform electric field in the context of new type of the extended uncertainty principle and the energy spectrum and the corresponding eigenfunctions are extracted.
- In chapter 3, the three-dimensional K-G oscillator and the K-G equation with a Coulomb plus scalar potential are treated in the context of Snyder-de Sitter algebra, the energy spectrum and the corresponding wave functions are calculated and the particular cases are deduced, such as the case of the absence of deformation and the case of the zero electric field.

-The second part is devoted especially to the Feynman's approach, we set up a supersymmetric path integral formulation in the context of the EUP to establish the Green function for the Dirac oscillator problem. Following the global representation for the causal Green function is obtained and the Schwinger proper-time method is introduced . To determine the appropriate quantum fluctuations and avoid any ambiguities, we discretize the measure and choose for any δ-point discretization interval. With the aid of appropriate transformations, the propagator has converted to the case of the standard problem of the Poschl-Teller potential. We obtained the energy spectrum and the corresponding wave functions then we deduced also the special cases are considered . The last chapter is devoted to a summary of the main findings and general conclusions.

Chapitre 2

Resolution of some relativistic problems in the context of new type of the extended uncertainty principle :

2.1 Introduction

Besides of certain different forms of the deformed algebras models mentioned in the introduction such as the case of generalized uncertainty principle, the (anti) -de Sitter background associated to the topology of the physical space or the extended uncertainty principle etc. ...., another deformed form has emerged in the literature of various areas during these last years, known by the name the new type of EUP with a minimum momentum dispersion[53, 54, 55]. This new type of EUP has been introduced by the action of the translation operator in a space with a diagonal metric for the purpose of describing the motion of a quantum particle in the curved space.

$$T_\gamma(\delta x) \mid x >=\mid x + \delta x + \gamma x \delta x > \tag{2.1}$$

where δx is an infinitesimal displacement and the parameter γ is the inverse of a characteristic length that determines the mixing between the displacement and the original position state[53, 54]. This translation is non-additive, can be written as to first order in δx

$$T_\gamma(\delta x) = 1 - \frac{i\delta x}{\hbar} P_\gamma. \tag{2.2}$$

where P_γ is a generalized momentum operator. This property changes the commutation relation for position and momentum as

$$[\hat{x}, P_\gamma] = i\hbar(1 + \gamma x), \tag{2.3}$$

and leads a generalized uncertainty relation

$$\Delta x \Delta P_\gamma \geqslant \frac{\hbar}{2}(1 + \gamma \langle x \rangle). \tag{2.4}$$

The generalized momentum operator and the operators of position satisfying equation (2.3) can be represented in Hermitian form by [53, 54]

$$P_\gamma = -i\hbar D_\gamma \quad \text{and} \quad \hat{x} = x, \tag{2.5}$$

with

$$D_\gamma = \left[(1 + \gamma x)\frac{d}{dx} + \frac{\gamma}{2}\right] \tag{2.6}$$

On the other hand, the nonadditive operator corresponds to the infinitesimal generator of the q-exponential function[85]

$$\exp_q(x) \equiv [1 + (1-q)x]^{\frac{1}{1-q}}, \tag{2.7}$$

where x is a dimensionless variable and $\lambda \equiv (1-q)$. The equation (2.7) represents a fundamental mathematical definition for the generalized thermostatistics of Tsallis and its applications [54]. For this purpose for see what kind of physical importance the translation operator bears within this framework, some problems were solved for a quantum system. For example, the study of a particle under a null potential confined in a square well [53, 54], the solution of the quantum harmonic oscillator where the problem is converted to the Morse potential case [86], the position-dependent mass system with a variable potential [87] and, Arda used this displacement operator to study the particle moving in an inverse square plus Coulomb-like potential which is similar to the Rosen-Morse potential in usual position space [88], a deformed Bohmian formalism by means of a deformed Fisher information functional and a derivation a deformed Cramer-Rao bound in [89], a displaced anisotropic two-dimensional non-Hermitian harmonic oscillator and graphics for the specific heat and for the entropy of both oscillators compared with several experimental by [90], the classical mechanics in the curved space and Bohr-Sommerfeld quantization [55] and a particle confined in a bidimensional box within a generalized space [91].

The main purpose is to solve analytically and exactly in the context of this new type of EUP for some important applications :

-Klein Gordon particle in a box model.

-Klein Gordon equation with linear vector and scalar potentials.

-Klein Gordon equation with inversely linear vector and scalar potentials of Coulomb-type.

-Klein-Gordon and Dirac oscillators with a uniform electric field using the displacement operator method.

Consequently, our attempt is to approach this new type of EUP for a relativistic problem and to see the influence of this deformation on the properties of the systems such as the confinement phenomenon and energy value of the Stark shift.

2.2 Klein Gordon particle in a one dimensional box case :

We consider a Klein-Gordon (K-G) particle without spin of mass m and charge q confined to the following one dimensional box :

$$qV(x) = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & elsewhere \end{cases} . \tag{2.8}$$

So, in the context of this new type of EUP using the displacement operator method, the stationary Klein-Gordon equation in the presence of a potential $V(x)$ in one dimensional space is defined by : we put $(\hbar = c = 1)$

$$\left[(E - qV(x))^2 - P_\gamma^2 - m^2\right] \phi(x) = 0, \tag{2.9}$$

where P_γ is given by (2.5) . Moreover, the continuity equation can be deduced from the modified Klein-Gordon Eq (2.9) and its conjugate by this relation

$$\frac{\partial \rho}{\partial t} + D_\gamma J_\gamma = 0, \tag{2.10}$$

with

$$\rho = \frac{i}{2m} \left(\Psi^* \partial_t \Psi - \Psi \partial_t \Psi^*\right), \tag{2.11}$$

and J_γ defines the modified current density

$$J_\gamma = -\frac{i}{2m} \left(\Psi^* (1 + \gamma x) \frac{d\Psi}{dx} - \Psi (1 + \gamma x) \frac{d\Psi^*}{dx}\right). \tag{2.12}$$

Now, in order to solve the equation (2.9) in one dimensional box, for $0 \leq x \leq L$, using the representation (2.5) and the following transformation :

$$u = (1 + \gamma x), \tag{2.13}$$

we obtain :

$$\left(u^2\frac{d^2}{du^2}+2u\frac{d}{du}+\frac{1}{4}+\frac{E^2-m^2}{\gamma^2}\right)\phi\left(u\right)=0. \tag{2.14}$$

To transform this last differential equation homogeneous to another one with constant coefficients, using the following change $z=\ln u$, we get as a result :

$$\left(\frac{d^2}{dz^2}+\frac{d}{dz}+\frac{1}{4}+\frac{E^2-m^2}{\gamma^2}\right)\phi\left(z\right)=0, \tag{2.15}$$

whose the solution in term on the old variable is given by

$$\phi\left(x\right)=\frac{\mathcal{N}}{\sqrt{(1+\gamma x)}}\sin\left(\sqrt{E^2-m^2}\frac{\ln\left(1+\gamma x\right)}{\gamma}+\xi\right), \tag{2.16}$$

where $\mathcal{N}$ is a normalization constant. Using the boundary conditions $\varphi\left(0\right)=\varphi\left(L\right)=0$, the solution of (2.14) will take the following form

$$\phi\left(x\right)=\frac{\mathcal{N}}{\sqrt{(1+\gamma x)}}\sin\left(\sqrt{E_n^2-m^2}\frac{\ln\left(1+\gamma x\right)}{\gamma}\right), \tag{2.17}$$

with

$$\sqrt{E_n^2-m^2}\frac{\ln\left(1+\gamma L\right)}{\gamma}=n\pi. \tag{2.18}$$

This gives rise to the quantized energy

$$E_n^{\pm}=\pm\sqrt{m^2+\frac{n^2\pi^2\gamma^2}{\ln^2\left(1+\gamma L\right)}}. \tag{2.19}$$

Now, if we consider $\gamma=0$ absence of deformation, taking $\gamma\rightarrow 0$ in (2.19) we find,

$$E^{\pm}=\pm\sqrt{m^2+\frac{n^2\pi^2}{L^2}}, \tag{2.20}$$

which is the result of the ordinary case[92].

The normalization constant $\mathcal{N}$ can be obtained from the normalization condition of the Ψ_n, follows from the modified definition of the scalar product for Klein Gordon equation :

$$\int_{-\infty}^{+\infty}\frac{i}{2m}\left(\Psi_n^*\left(x\right)\frac{\partial\Psi_n\left(x\right)}{\partial t}-\Psi_n\left(x\right)\frac{\partial\Psi_n^*\left(x\right)}{\partial t}\right)=1, \tag{2.21}$$

and by a direct calculation, we get

$$\mathcal{N}=\sqrt{\frac{2\gamma m}{E_n\ln\left(1+\gamma L\right)}}. \tag{2.22}$$

2.3 Klein Gordon equation with mixed scalar and vector linear potentials case :

The dynamic of Klein-Gordon particle in $(1+1)$ dimension in the presence of a scalar potential $S(x)$ and a vector potential $V(x)$ in the framework of of new type of EUP is governed by this stationary equation :

$$\left[P_\gamma^2+(m+S(x))^2-\left(E-qV\left(x\right)\right)^2\right]\psi\left(x\right), \tag{2.23}$$

where the vector and the scalar potential are chosen linear as follows

$$qV(x)=V_0x \tag{2.24}$$
$$S(x)=S_0x.$$

and we take $S_0^2-V_0^2>0$ so as to avoid complex eigenvalues. We replace $S(x)$ and $V(x)$ and using the representation (2.5) and (2.24), the equation (2.23) becomes :

$$\left[\frac{d^2}{du^2}+\frac{2}{u}\frac{d}{du}+\frac{A}{u^2}+\frac{B}{u}-C^2\right]\psi\left(u\right)=0, \tag{2.25}$$

where we have used the same transformation (2.13) and this notation

$$A = \frac{V_0^2 - S_0^2}{\gamma^4} + \frac{2(EV_0 + mS_0)}{\gamma^3} + \frac{E^2 - m^2}{\gamma^2} + \frac{1}{4}$$
$$B = \frac{2(S_0^2 - V_0^2)}{\gamma^4} - \frac{2(EV_0 + mS_0)}{\gamma^3},$$
$$C = \frac{\sqrt{S_0^2 - V_0^2}}{\gamma^2}\ . \tag{2.26}$$

To simplify the equation (2.25), we introduce,

$$\psi(u) = u^\sigma \exp(-Cu) F(u), \tag{2.27}$$
$$u \mapsto y = 2Cu,$$

so, the differential equation will reduce to the equation of the associated Laguerre polynomials $L_n^k(y)$,

$$\left[y\frac{d^2}{dy^2} + [(2\sigma + 2) - y]\frac{d}{dy} + \frac{1}{y}[\sigma(\sigma - 1) + 2\sigma + A] + \frac{1}{2C}[B - 2C - 2C\sigma]\right] F(y) = 0. \tag{2.28}$$

by imposing the constraint,

$$\sigma(\sigma - 1) + 2\sigma + A = 0, \tag{2.29}$$

to eliminate the coefficient proportional to $\frac{1}{y}$, and

$$\begin{cases} \frac{1}{2C}[B - 2C - 2C\sigma] = n, \\ 2\sigma + 2 = k + 1. \end{cases} \tag{2.30}$$

The relation (2.29) leads to the following expressions for σ by

$$\sigma_\pm = -\frac{1}{2} \pm \frac{1}{\gamma}\sqrt{\left(m + E - \frac{(S_0 - V_0)}{\gamma}\right)\left(m - E - \frac{(S_0 + V_0)}{\gamma}\right)} \tag{2.31}$$

Among these two solutions, the physically acceptable one is only σ_+. To extract the energy spectrum, we substitute the expression (2.31) into the first relation of (2.30), then it is straightforward to show that

$$E^{\pm}=-\frac{mV_0}{S_0}-\gamma\frac{V_0\sqrt{S_0^2-V_0^2}}{S_0^2}\left(n+\frac{1}{2}\right)\pm\frac{S_0^2-V_0^2}{S_0^2}\sqrt{-\gamma^2\left(n+\frac{1}{2}\right)^2-\frac{\gamma\left(2n+1\right)mS_0}{\sqrt{S_0^2-V_0^2}}+\frac{\left(2n+1\right)S_0^2}{\sqrt{S_0^2-V_0^2}}}, \tag{2.32}$$

It is remarkable that the expression of the energy spectrum is a dependent function of the deformation parameter γ, γ^2 and with powers in n, n^2 which explains the phenomenon of confinement due to the new type of extended uncertainty principle. Moreover , for large values of n, the second term is not defined of $E^{\pm}$. In order to ensure the positivity of the square root of energy, one must impose an upper bound on the allowed values of n.

Solving the equation (2.23) along with (2.27), (2.28) and (2.31), we obtain the final form of the wave function in the former variable x as

$$\psi(x)=N_{n\lambda}\left(1+\gamma x\right)^{-\frac{1}{2}+\frac{1}{\gamma}\sqrt{\left(m+E-\frac{(S_0-V_0)}{\gamma}\right)\left(m-E-\frac{(S_0+V_0)}{\gamma}\right)}}\exp\left\{-\frac{\sqrt{(S_0^2-V_0^2)}}{\gamma^2}\left(1+\gamma x\right)\right\}$$

$$\bullet L_n^{\frac{2}{\gamma}\sqrt{\left(m+E-\frac{(S_0-V_0)}{\gamma}\right)\left(m-E-\frac{(S_0+V_0)}{\gamma}\right)}}\left(\frac{2\sqrt{(S_0^2-V_0^2)}}{\gamma^2}(1+\gamma x)\right), \tag{2.33}$$

and N_{nr} is a normalization constant.

Now if we consider $\gamma=0$ absence of deformation, we replace $\gamma=0$ in (2.32) we find,

$$E^{\pm}=-\frac{mV_0}{S_0}\pm\frac{(S_0^2-V_0^2)^{\frac{3}{4}}}{S_0}\sqrt{(2n+1)}, \tag{2.34}$$

which is the result of the ordinary case[95, 96]

2.4 Klein Gordon equation with mixed scalar and vector inversely linear potentials case :

In this case we choose the vector and the scalar potential inversely linear of Coulomb-type as follows

$$\begin{aligned} qV(x) &= \frac{V_0}{\mid x \mid} \\ S(x) &= \frac{S_0}{\mid x \mid}, \end{aligned} \tag{2.35}$$

Using the transformation $u = 1 + \gamma \mid x \mid$ and the representation (2.5), for $x > 0$, the stationary Klein-Gordon equation in $(1 + 1)$ dimension in the framework of of new type of EUP (2.23) can be written as :

$$\left[\frac{d^2}{du^2} + \frac{2}{u}\frac{d}{du} + \frac{a_1}{u^2} + \frac{a_2}{u(1-u)} - \frac{a_3^2}{(1-u)^2}\right]\Psi(u) = 0 \tag{2.36}$$

where we replaced $S(x)$ and $V(x)$ by their expressions (2.35) and this notation,

$$\begin{aligned} a_1 &= \frac{2(EV_0 + mS_0)}{\gamma} - (S_0^2 - V_0^2) + \frac{(E^2 - m^2)}{\gamma^2} + \frac{1}{4}, \\ a_2 &= \frac{2(EV_0 + mS_0)}{\gamma} - 2(S_0^2 - V_0^2), \\ a_3 &= \sqrt{S_0^2 - V_0^2)}\ , \quad S_0 > V_0 \end{aligned} \tag{2.37}$$

In addition, we note that this Eq (2.36) possesses three singular points $0, 1, \infty$. By means of the substitution $\Psi(u) = u^p(1-u)^q\varphi(u)$, this equation will reduce to the hypergeometric type

$$\left[u(1-u)\frac{d^2}{du^2} + [(2p+2) - (2p+2q+2)u]\,\frac{d}{du} + [a_2 - 2pq - 2q]\right]\varphi(u) = 0. \tag{2.38}$$

where p and q are fixed as follows,

$$\begin{cases} p = -\frac{1}{2} \pm \sqrt{(S_0^2 - V_0^2) - \frac{2(EV_0+mS_0)}{\gamma} - \frac{(E^2-m^2)}{\gamma^2}} \\ q = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (S_0^2 - V_0^2)}, \end{cases} \tag{2.39}$$

and the solution of Eq. (2.38) can be written as

$$\varphi(u) \sim_2 F_1\left(a, b; c; u\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{u^k}{k!} \tag{2.40}$$

with the parameters a, b and c are given by

$$\begin{cases} a = p + q + \frac{1}{2} - i\sqrt{\frac{(E^2-m^2)}{\gamma^2}} \\ b = p + q + \frac{1}{2} + i\sqrt{\frac{(E^2-m^2)}{\gamma^2}} \\ c = 1 \pm 2\sqrt{(S_0^2 - V_0^2) - \frac{2(EV_0+mS_0)}{\gamma} - \frac{(E^2-m^2)}{\gamma^2}} \end{cases} \tag{2.41}$$

The mathematical solutions of Eq. (2.36) in the former variable x as

$$\Psi(x)) = N_\gamma (1 + \gamma x)^p x^q {}_2F_1\left(a, b; c; 1 + \gamma x\right), \tag{2.42}$$

where N_γ is the normalization constant and the boundary condition that ($u \longrightarrow 1$or $x \longrightarrow 0$) leads the wave function tending to finite, the hypergeometric function reduced to a polynomial with the following restriction

$$a = -n, \tag{2.43}$$

which is the quantization rule of the system and gives us the energy eigenvalues as

$$E_n^{\pm} = -V_0 \frac{mS_0 + \frac{\gamma}{2}\left[(n+q)^2 - (S_0^2 - V_0^2)\right]}{V_0^2 + (n+q)^2}$$

$$\pm\frac{1}{2}\left\{\frac{V_0^2\left[2\gamma m S_0 + \gamma^2\left((n+q)^2 + (V_0^2 - S_0^2)\right)\right]^2}{\left[V_0^2 + (n+q)^2\right]^2} + \right.$$

$$\left.\frac{4m^2\left[(n+q)^2-S_0^2\right]-4\gamma m S_0\left[(n+q)^2-(S_0^2-V_0^2)\right]-\gamma^2\left[(n+q)^2-(S_0^2-V_0^2)\right]^2}{V_0^2+(n+q)^2}\right\}^{\frac{1}{2}} \tag{2.44}$$

Also for this case, for large values of n, the second term is not defined . In order to ensure the positivity of the square root of energy, one must impose an upper bound on the allowed values of n.

Now in our analysis, it is interesting to study two particular cases

First ; if $\gamma = 0$ absence of deformation, we replace $\gamma = 0$ in (2.44) we find,

$$E_n^{\pm} = \frac{-mV_0S_0}{V_0^2+(n+q)^2} \pm m\sqrt{\frac{(n+q)^2-S_0^2}{V_0^2+(n+q)^2}} \tag{2.45}$$

Second, if $\gamma = 0$ and $S_0 = 0$, taking $(\gamma \to 0)$ and $S_0 = 0$,the expression of energy spectrum (2.44) become

$$E_n^{\pm} = \pm\frac{m}{\sqrt{1+\frac{V_0^2}{(n+q)^2}}} \tag{2.46}$$

which coincides exactly with those of the literatures[97].

At the end of this section, we mention that in the region $x < 0$, we get the same form of the solution (2.42)if we make the change of the variables $y = -x$.

In this contribution, we have established an exact and explicit solution of some problems in the context of new type of the extended uncertainty principle using the displacement operator method such as : The Klein-Gordon particle confined in a one dimensional box, the scalar particle with linear vector and scalar potentials and the case of Coulomb-type vector and scalar potentials. In these three cases, the exact analytical solution is determined, the wave functions and the exact energy spectrum are obtained depending on the deformation parameter γ. On the other hand, the expressions of energy spectrum vary with all the power of n, which explain the confinement phenomenon. Also, it is mentioned that for the last two cases, bound states are limited, the expressions of energy are not defined for large values of n, one must impose an upper bound on the allowed values of n. Finally the limiting cases are presented.

2.5 Klein-Gordon oscillator equation with a uniform electric field

In regular space, the Klein-Gordon oscillator subject to an electric field Θ_{KG} in one dimensional space is defined by,

$$\Theta_{KG}\psi(x) = \left[(\hat{p} + im\omega\hat{x})(\hat{p} - im\omega\hat{x}) + m^2 - (E - q\varepsilon\hat{x})^2\right]\psi(x) = 0, \qquad (2.47)$$

which can be written as

$$\left\{p^2 + (m^2\omega^2 - \varepsilon^2)x^2 + im\omega[x,p] + 2\varepsilon Ex - (E^2 - m^2)\right\}\psi(x) = 0, \qquad (2.48)$$

where q is the electrical charge and ε is the intensity of electric field. Note that we use the units where $\hbar = c = 1$.

In order to solve the Eq. (2.48), we use the transformation (2.13) and using the representation (2.5) and (2.3), the Eq (2.48) becomes :

$$\left\{\frac{d^2}{du^2} + \frac{2}{u}\frac{d}{du} + \left(\frac{1}{4} - \frac{(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{2\varepsilon E}{\gamma^3} + \frac{(E^2 - m^2)}{\gamma^2}\right)\frac{1}{u^2}\right.$$
$$\left. + \left(\frac{2(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{m\omega}{\gamma^2} - \frac{2\varepsilon E}{\gamma^3}\right)\frac{1}{u} + \frac{(\varepsilon^2 - m^2\omega^2)}{\gamma^4}\right\}\psi(u) = 0. \qquad (2.49)$$

Introducing the notation

$$\begin{aligned}\delta &= \frac{1}{4} - \frac{(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{2\varepsilon E}{\gamma^3} + \frac{(E^2 - m^2)}{\gamma^2}, \\ \eta &= \left(\frac{2(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{m\omega}{\gamma^2} - \frac{2\varepsilon E}{\gamma^3}\right), \\ \zeta &= \frac{\sqrt{(m^2\omega^2 - \varepsilon^2)}}{\gamma^2} \text{ with } m\omega > \varepsilon,\end{aligned} \tag{2.50}$$

we get

$$\psi'' + \frac{2}{u}\psi' + \left(\frac{\delta}{u^2} + \frac{\eta}{u} - \zeta^2\right)\psi = 0 \tag{2.51}$$

To simplify Eq. (2.51), we introduce,

$$\begin{aligned}\psi(u) &= u^{\sigma} \exp(-\zeta u) F(u), \\ u &\mapsto y = 2\zeta u,\end{aligned} \tag{2.52}$$

where σ is a constant to be determined later. After using (2.52), the differential Eq. (2.51) will reduce to the equation of the associated Laguerre polynomials $L_n^k(y)$,

$$\left[y\frac{d^2}{dy^2} + [(2\sigma + 2) - y]\frac{d}{dy} + \frac{1}{y}[\sigma(\sigma - 1) + 2\sigma + \delta] + \frac{1}{2\zeta}[\eta - 2\zeta - 2\zeta\sigma]\right] F(y) = 0. \tag{2.53}$$

We impose the constraint,

$$\sigma(\sigma - 1) + 2\sigma + \delta = 0, \tag{2.54}$$

to eliminate the coefficient proportional to $\frac{1}{y}$, and

$$\begin{cases} \frac{1}{2\zeta}[\eta - 2\zeta - 2\zeta\sigma] = n, \\ 2\sigma + 2 = k + 1. \end{cases} \tag{2.55}$$

The relation (2.54) leads to the following expressions for σ by

$$\sigma_{\pm} = -\frac{1}{2} \pm \frac{1}{\gamma}\sqrt{m^2 - E^2 + \frac{m^2\omega^2 - \varepsilon^2}{\gamma^2} - \frac{2E\varepsilon}{\gamma}} \tag{2.56}$$

Among these two solutions, the physically acceptable one is only σ_+. To extract the energy spectrum, we substitute the expression (2.56) into the first relation of (2.55). Then it is straightforward to show that

$$E^{\pm} = -\frac{\varepsilon\gamma}{2m\omega}\left[(2n+1)\,\Omega - 1\right] \pm \Omega\sqrt{m^2 + m\omega\left[(2n+1)\Omega - 1\right] - \frac{\gamma^2}{4}\left[(2n+1)\,\Omega - 1\right]^2}, \tag{2.57}$$

with $\Omega = \frac{\sqrt{(m^2\omega^2 - \varepsilon^2)}}{m\omega}$. We should note that the expression of energy spectrum contains all corrections of all orders of $(\varepsilon\gamma)^2$. This is related to the exact contribution to the Stark effect in this framework of the deformation. On the other hand, it varies by the power of n^2, which explain the confinement phenomenon. For large values of n, the square of the energy spectrum $(E)^2$ becomes negative. Thus, in order to ensure the positivity of the the square of the energy, one must impose an upper bound on the allowed values of n.

Expanding up to the first order in γ^2, we obtain

$$\mathrm{E}^{\pm} = \pm\Omega\sqrt{m^2 + m\omega\left[(2n+1)\Omega - 1\right]} - \frac{\varepsilon\gamma}{2m\omega}\left[(2n+1)\,\Omega - 1\right] \mp \frac{\gamma^2\Omega\left[(2n+1)\,\Omega - 1\right]^2}{8\sqrt{m^2 + m\omega\left[(2n+1)\Omega - 1\right]}}. \tag{2.58}$$

The first term in (2.58) is the energy spectrum of the usual Klein-Gordon oscillator subject to the uniform electric field. The second and the third terms represent the quantum fluctuations due to the new type of extended uncertainty principle. It is remarkable that the expression of the energy spectrum contains additional deformed correction terms depending on the deformation parameter γ, γ^2 and with powers in n^2 which explains the phenomenon of confinement. We can see that the energy spectrum in the context of this deformation is smaller than the energy in the ordinary case.

Solving the Eq (2.47) along with the relations (2.52), (2.53) and (2.56), we obtain the final form of the wave function in the former variable x as

$$\psi(x) = N_{nr}\left(1+\gamma x\right)^{-\frac{1}{2}+\frac{1}{\gamma}\sqrt{m^2-E^2+\frac{m^2\omega^2-\varepsilon^2}{\gamma^2}-\frac{2E\varepsilon}{\gamma}}} \exp\left\{-\frac{1}{\gamma^2}\sqrt{(m^2\omega^2-\varepsilon^2)}\left(1+\gamma x\right)\right\}$$

$$L_n^{\frac{2}{\gamma}\sqrt{m^2-E^2+\frac{m^2\omega^2-\varepsilon^2}{\gamma^2}-\frac{2E\varepsilon}{\gamma}}}\left(\frac{2}{\gamma^2}\sqrt{(m^2\omega^2-\varepsilon^2)}(1+\gamma x)\right), \qquad (2.59)$$

and N_{nr} is a normalization constant.

We can present our results graphically for some numerical values of the physical parameters. We will take $m = 1$ and $\omega = 10$ in our analysis. We will plot the curves only for E^+ as the curves for E^- do not show different physical behavior.

In Fig.(1), we plot the energy levels as a function of quantum number n for various values of γ and for $\varepsilon = 0$. We see that the values for non-zero γ coincide. If we take a fixed but non-zero ε as in Fig.(2), we find that the energy behavior is different. The non-zero electric field yields a physical effect on the system. The Fig.(3) and Fig.(4) show the behavior of the energy for varying γ and for a fixed ε (we used $\varepsilon = 0$ and $\varepsilon = 9$, respectively). Here, we see the effect of γ on the energy behavior for some fixed n values.

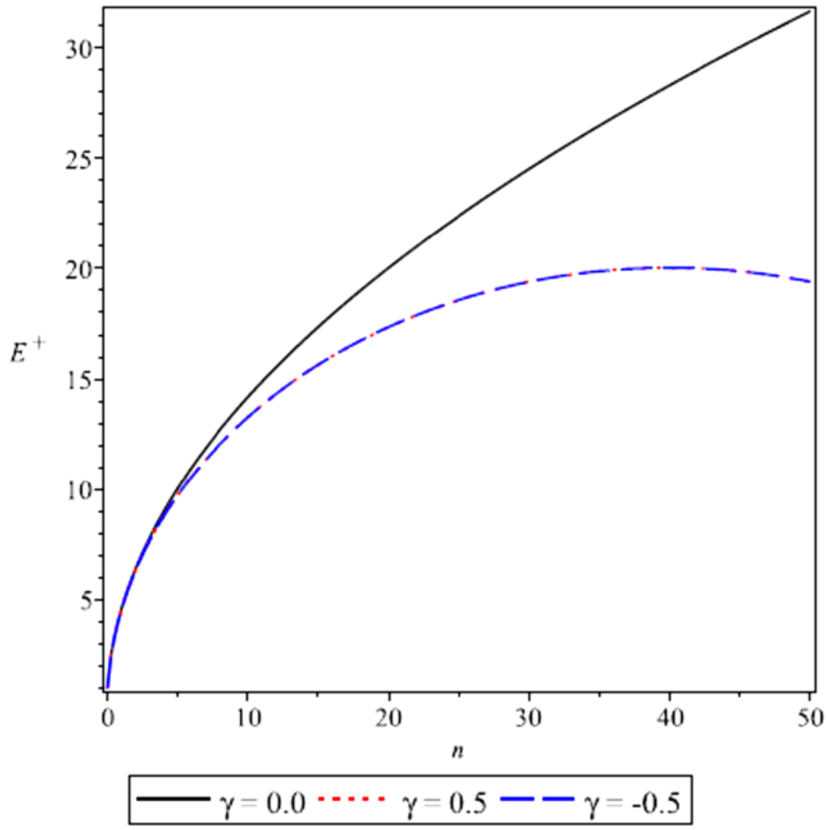

FIG. 1: (E^+ vs. n) for $\varepsilon = 0$ (Klein-Gordon Oscillator).

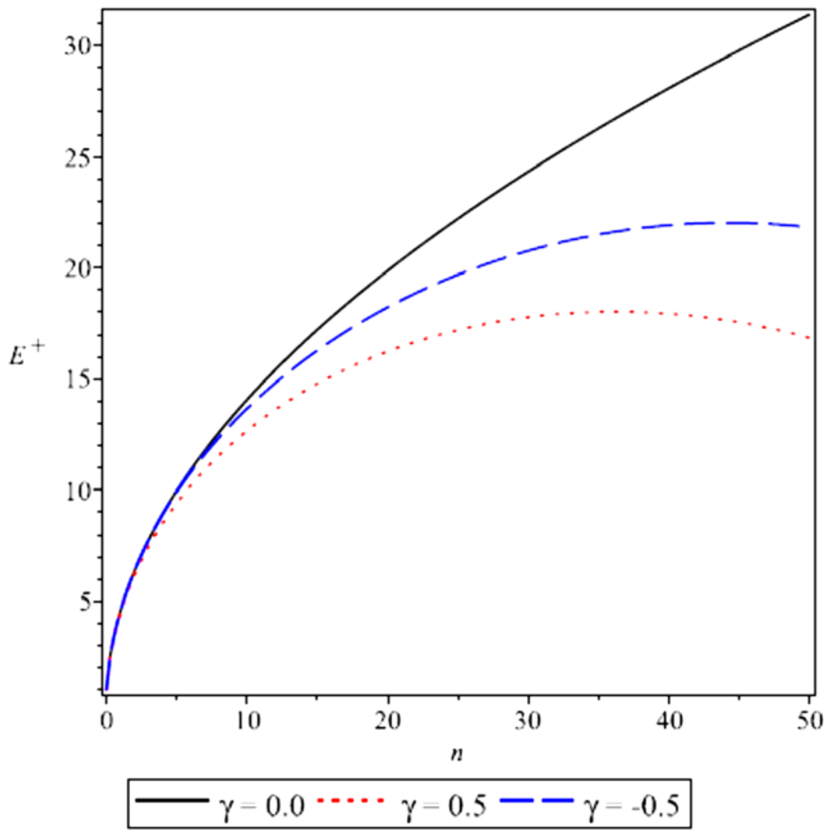

FIG. 2: (E^+ vs. n) for $\varepsilon = 1$ (Klein-Gordon Oscillator).

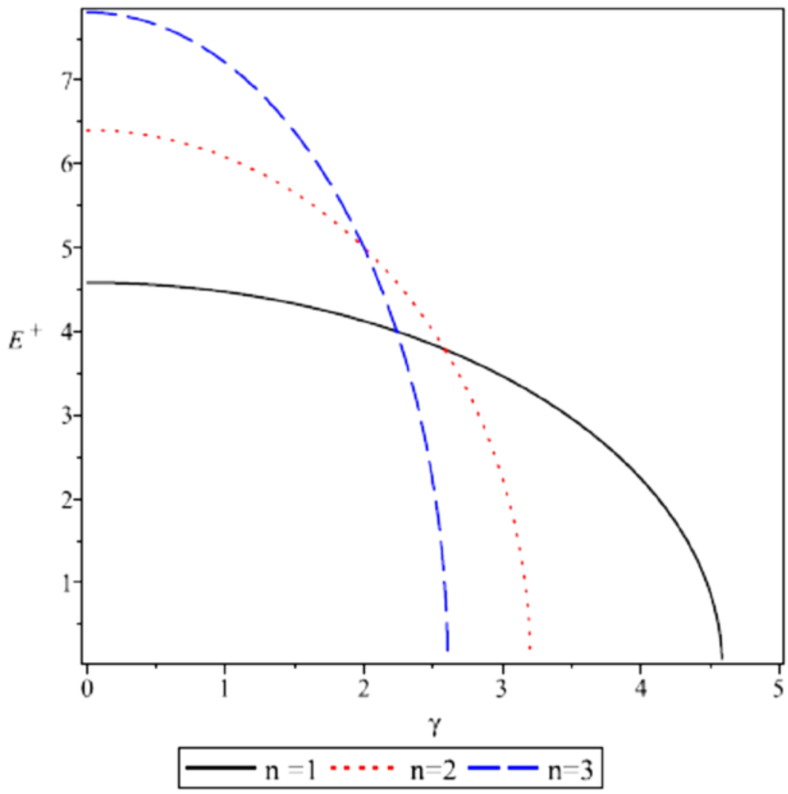

FIG. 3: (E^+ vs. γ) for $\varepsilon = 0$ (Klein-Gordon Oscillator).

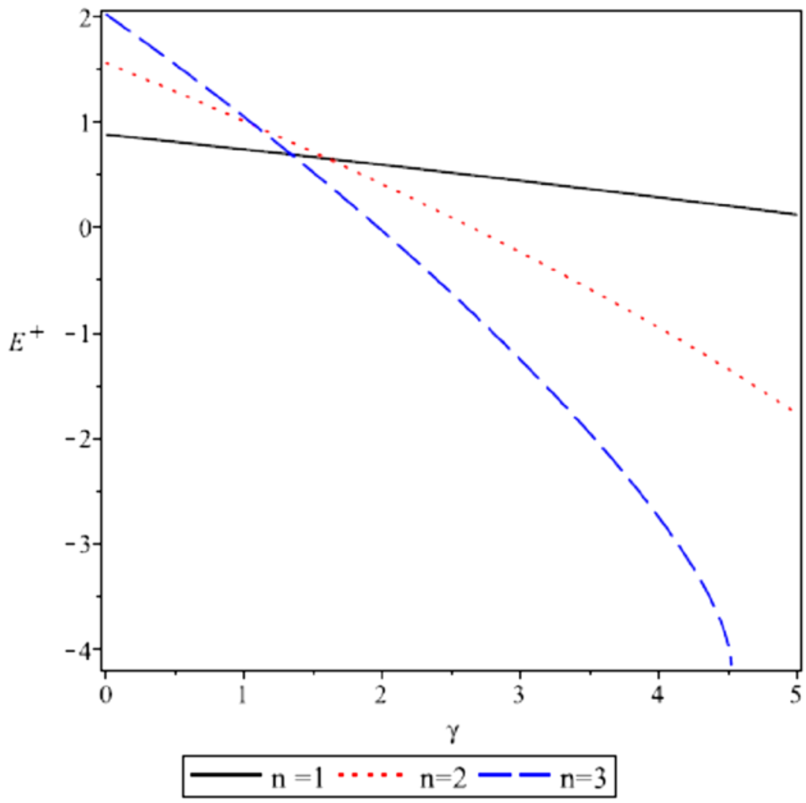

FIG. 4: (E^+ vs. γ) for $\varepsilon = 9$ (Klein-Gordon Oscillator).

Some special cases

We can consider some special cases for vanishing γ and ε.

For $\gamma = 0$, namely in the absence of deformation, we replace $\gamma = 0$ in (2.57) find,

$$E^{\pm} = \mp\Omega\sqrt{m^2 - m\omega + (2n+1)m\omega\Omega}. \tag{2.60}$$

The case for $\varepsilon = 0$, namely in the absence of an electric field implies $\Omega = 1$, and the expression of the energy spectrum (2.57) becomes

$$E^{\pm} = \pm\sqrt{-\gamma^2 n^2 + m^2 + 2nm\omega}. \tag{2.61}$$

In the case where $\gamma = \varepsilon = 0$, we have the pure Klein-Gordon oscillator case. This limit yields

$$E^{\pm} = \pm\sqrt{m^2 + 2nm\omega}, \tag{2.62}$$

which is in agreement with the result of the ordinary case.

2.6 Dirac oscillator equation with a uniform electric field

The Dirac oscillator with a uniform electric field is defined by the expression [98],

$$\left[\alpha\left(\hat{p} - im\omega\beta\hat{x}\right) + \beta m\right]\Psi(x) = \left(E - q\varepsilon\hat{x}\right)\Psi(x), \tag{2.63}$$

where $\Psi(x) = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$ and α, β are the Dirac matrices given by

$$\alpha = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{and} \quad \beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.64}$$

Note that we are using the units where $(\hbar = c = 1)$. Using the matrices (2.64) and the definition of $\Psi(x)$ in Eq (2.63), we obtain the system,

$$\begin{cases} (p_x + im\omega x)\,\phi_2 = (E - m - \varepsilon x)\phi_1(x)\,, \\ (p_x - im\omega x)\,\phi_1 = (E + m - \varepsilon x)\phi_2(x)\,. \end{cases} \tag{2.65}$$

Introducing the notation $\Pi^{\pm} = p_x \pm im\omega x$ and $M^{\pm} = E \pm m - \varepsilon x$, the new form of the system (2.65) can be obtained as

$$\begin{cases} \Pi^{+}\phi_2(x) = M^{-}\phi_1(x)\,, \\ \Pi^{-}\phi_1(x) = M^{+}\phi_2(x)\,. \end{cases} \tag{2.66}$$

In order to decouple the above system, we write ϕ_2 in terms of ϕ_1,

$$\phi_2(x) = \left(M^{+}\right)^{-1}\Pi^{-}\phi_1(x)\,, \tag{2.67}$$

and we replace it in the first equation as

$$\Pi^{+}\left(M^{+}\right)^{-1}\Pi^{-}\phi_1(x) = M^{-}\phi_1(x)\,,$$

using

$$\Pi^{+}\left(M^{+}\right)^{-1} = \left(M^{+}\right)^{-1}\Pi^{+} + [\Pi^{+}, \left(M^{+}\right)^{-1}]. \tag{2.68}$$

Then we multiply the whole equation by M^{+} on the left to get

$$\left[\Pi^{+}\Pi^{-} - M^{+}M^{-} + M^{+}[\Pi^{+}, \left(M^{+}\right)^{-1}]\Pi^{-}\right]\phi_1(x) = 0, \tag{2.69}$$

where $[\cdot,\cdot]$ is the commutator between two operators.

We notice that the first two terms represent exactly the Klein-Gordon oscillator. We use

$$\Theta_{KG} = \Pi^{+}\Pi^{-} - M^{+}M^{-}, \tag{2.70}$$

where

$$\Theta_{KG} = \ (\hat{p} + im\omega\hat{x})\,(\hat{p} - im\omega\hat{x}) + m^2 - (E - q\varepsilon\hat{x})^2\,, \tag{2.71}$$

and the third term characterizes the spinor effect of the particle. Using the definitions, the equation (2.69) can be written as

$$\left\{\Theta_{KG} + (E+m-\varepsilon x)\left[(p_x + im\omega x), \frac{1}{(E+m-\varepsilon x)}\right](p_x - im\omega x)\right\}\phi_1(x) = 0. \tag{2.72}$$

By a direct calculation, the Eq (2.72) becomes

$$\left\{\Theta_{KG} - \frac{i\varepsilon(1+\gamma x)}{(E+m-\varepsilon x)}(p_x - im\omega x)\right\}\phi_1(x) = 0, \tag{2.73}$$

where we used the Eq (2.3).

To solve the Eq. (2.73), we use the change of variable (2.13). Then we obtain,

$$\left\{\frac{d^2}{du^2} + \left(\frac{2}{u} + \frac{1}{r-u}\right)\frac{d}{du} + \frac{\eta}{u} + \frac{m\omega}{\gamma^2}\frac{1}{(r-u)} + \frac{\delta}{u^2} + \frac{\tau}{u(r-u)} - \zeta^2\right\}\phi_1(u) = 0, \tag{2.74}$$

where

$$\begin{aligned}
\delta &= \frac{1}{4} - \frac{(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{2\varepsilon E}{\gamma^3} + \frac{(E^2 - m^2)}{\gamma^2}, \\
\eta &= \left(\frac{2(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{m\omega}{\gamma^2} - \frac{2\varepsilon E}{\gamma^3}\right), \\
\tau &= \left(\frac{1}{2} - \frac{m\omega}{\gamma^2}\right), \\
\zeta^2 &= \frac{\sqrt{(m^2\omega^2 - \varepsilon^2)}}{\gamma^2} \quad \text{with} \;\; m\omega > \varepsilon, \\
r &= \frac{\gamma\,(E+m)}{\varepsilon} + 1.
\end{aligned} \tag{2.75}$$

In order to simplify the Eq (2.74), we introduce

$$\phi_1(u) = u^{\sigma}\exp(-\zeta u)\,G(u)\,, \tag{2.76}$$

where σ is a constant to be determined letter. We obtain

$$\left\{\frac{d^2}{du^2}+\left(\frac{2\sigma+2}{u}+\frac{1}{r-u}-2\zeta\right)\frac{d}{du}+\frac{1}{u^2}\left(\sigma(\sigma-1)+2\sigma+\delta\right)+\right.$$

$$\left.\frac{1}{u}\left(-2\zeta\sigma+\eta-2\zeta\right)+\frac{1}{(r-u)}\left(\frac{m\omega}{\gamma^2}-\zeta\right)+\frac{1}{u(r-u)}(\sigma+\tau)\right\}G(u)=0 \qquad (2.77)$$

To reduce this equation to a class of known differential equation with a polynomial solution, we need to to eliminate the coefficient proportional to $\frac{1}{u^2}$. We impose

$$\sigma(\sigma-1)+2\sigma+\delta=0, \qquad (2.78)$$

and this leads to the expression

$$\sigma_{\pm}=-\frac{1}{2}\pm\frac{1}{\gamma}\sqrt{m^2-E^2+\frac{m^2\omega^2-\varepsilon^2}{\gamma^2}-\frac{2E\varepsilon}{\gamma}}. \qquad (2.79)$$

Among these two solutions, the physically acceptable one is only σ_+, and the second solution leads to a non-physical wave function. We introduce $z=\frac{u}{r}$, then Eq. (2.77) takes the form

$$\left\{\frac{d^2}{dz^2}+\left(\frac{2\sigma+2}{z}-\frac{1}{z-1}-2r\zeta\right)\frac{d}{dz}+\frac{(-2r\zeta\sigma+r\eta-2r\zeta+\sigma+\tau)}{z}+\frac{(-\frac{rm\omega}{\gamma^2}+r\zeta-\sigma-\tau)}{z-1}\right\}\mathrm{G}(z)\mathrm{=}0, \qquad (2.80)$$

which is the confluent Heun differential equation [93, 94]. Let us denote the confluent Heun function by H_C, then the solutions can be written as

$$G(z)=C_1H_C(a,b,c,d,e,z)+C_2\exp(b)H_C(a,-b,c,d,e,z) \qquad (2.81)$$

with

$$a=-2\left(\frac{\gamma\left(E+m\right)}{\varepsilon}+1\right)\sqrt{\frac{(m^2\omega^2-\varepsilon^2)}{\gamma^4}},\tag{2.82}$$

$$b=\frac{2}{\gamma}\sqrt{m^2+\frac{m^2\omega^2}{\gamma^2}-\left(E+\frac{\varepsilon}{\gamma}\right)^2},$$

$$c=-2,$$

$$d=\left(\frac{\gamma\left(E+m\right)}{\varepsilon}+1\right)\left(\frac{2(m^2\omega^2-\varepsilon^2)}{\gamma^4}-\frac{2\varepsilon E}{\gamma^3}\right),$$

$$e=-\left(\frac{\gamma\left(E+m\right)}{\varepsilon}+1\right)\left(\frac{2(m^2\omega^2-\varepsilon^2)}{\gamma^4}+\frac{m\omega}{\gamma^2}-\frac{2\varepsilon E}{\gamma^3}\right)+\frac{m\omega}{\gamma^2}+1.$$

Then, the final expression for $\phi_1\left(x\right)$ is

$$\phi_1\left(x\right)=\left(1+\gamma x\right)^{\sigma}$$

$$\exp(\text{-}\zeta\left(1+\gamma x\right))\left[C_1H_C(a,b,c,d,e,\frac{(1+\gamma x)}{r})+C_2\exp(-b)H_C(a,-b,c,d,e,\frac{(1+\gamma x)}{r})\right]\tag{2.83}$$

Using the relation (2.67) and the expression of $\phi_1\left(x\right)$ we also find

$$\phi_2\left(x\right)=\frac{-i}{E+m-\varepsilon x}\left(\left(1+\lambda x\right)\frac{d}{dx}+\frac{\lambda}{2}+m\omega x\right)\phi_1\left(x\right)\tag{2.84}$$

In order to have a polynomial solution for the confluent Heun equation, we need to cut the series which are given by the recurrence relation. For a polynomial solution of degree N, we impose [93],

$$\frac{d}{a}+\frac{b+c}{2}+N+1=0.\tag{2.85}$$

Using the condition (2.85) and replacing the parameters a, b and c by their expressions (2.82), we finally get the following energy spectrum

$$E^{\pm}=-\varepsilon\gamma\frac{\Omega N}{m\omega}\pm\Omega\sqrt{m^2+2m\omega\Omega N-\gamma^2\Omega^2N^2}\quad\text{with}\qquad\Omega=\frac{\sqrt{(m^2\omega^2-\varepsilon^2)}}{m\omega}\tag{2.86}$$

In this case, one notes practically the same remarks of the Klein-Gordon oscillator case. The expression of the energy spectrum contains all corrections of all orders of $(\varepsilon\gamma)^2$. This is related with the exact contribution to the Stark effect in this deformation framework and it varies with the power of N^2, which explains the confinement phenomenon. For large values of N, the square of the energy spectrum $(E)^2$ becomes negative and, in order to ensure positivity of the the square of the energy, one must impose an upper bound on the allowed values of N.

Expanding the energy spectrum up to first order in γ^2, we obtain

$$E^{\pm} = \pm\Omega\sqrt{m^2 + 2m\omega\Omega N} - \varepsilon\gamma\frac{\Omega N}{m\omega} \mp \frac{\gamma^2\Omega^3 N^2}{2\sqrt{m^2 + 2m\omega\Omega N}}. \tag{2.87}$$

The first term in (2.87) is the energy spectrum of the usual Dirac oscillator subject to a uniform electric field. The second and the third terms represent the quantum fluctuations due to the new type of extended uncertainty principle.

We can also present our results for the Dirac oscillator graphically for some numerical values of the physical parameters. We will take $m = 1$ and $\omega = 10$ in our analysis. We will plot the curves only for E^+ as the curves for E^- do not show different physical behavior. One can easily see that the energy behavior is the same as in the Klein-Gordon oscillator case.

In Fig.(5), we plot the energy levels as a function of quantum number N for various values of γ and for $\varepsilon = 0$. We see that the values for non-zero γ coincide. If we take a fixed but non-zero ε as in Fig.(6), we find that the energy behavior is different. The non-zero electric field yields a physical effect on the system. The Fig.(7) and Fig.(8) show the behavior of the energy for varying γ and for a fixed ε (we used $\varepsilon = 0$ and $\varepsilon = 9$, respectively). Here, we see the effect of γ on the energy behavior for some fixed N values.

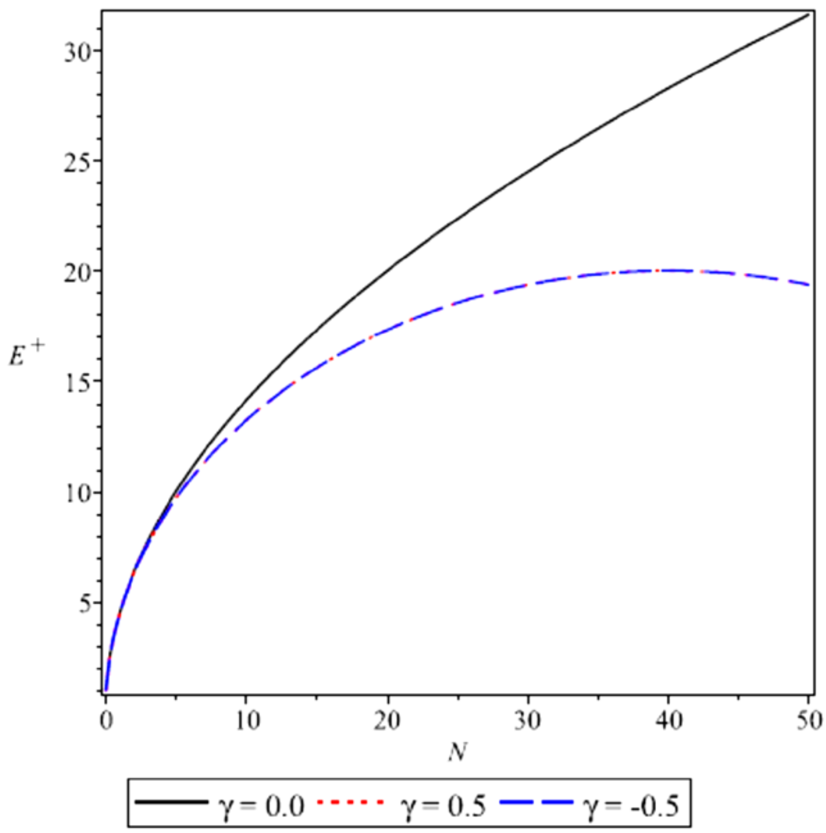

FIG. 5: (E^+ vs. n) for $\varepsilon = 0$ (Dirac Oscillator).

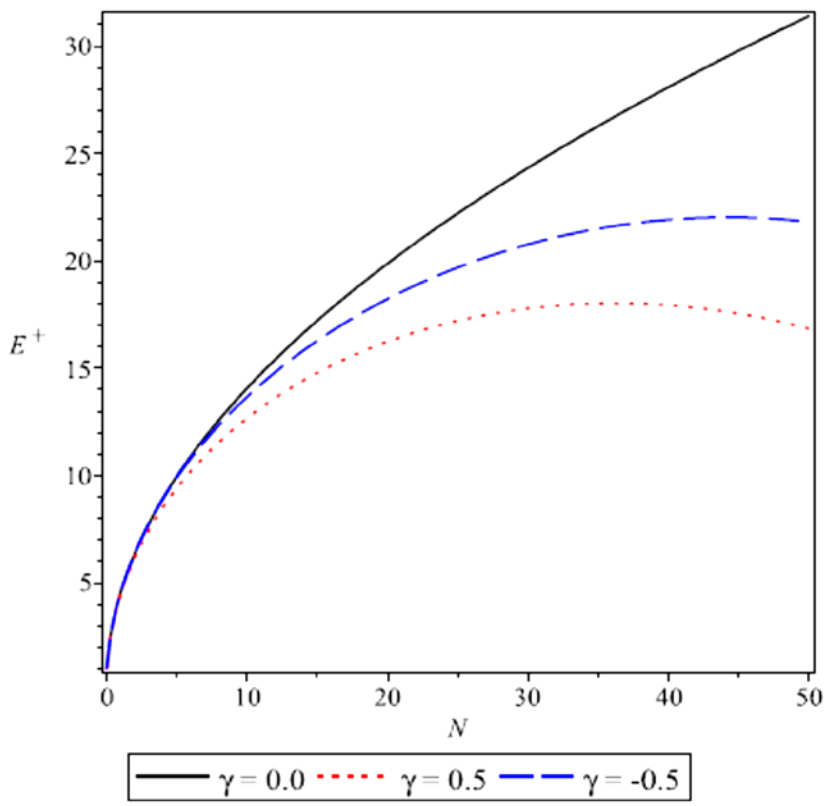

FIG. 6: (E^+ vs. n) for $\varepsilon = 1$ (Dirac Oscillator).

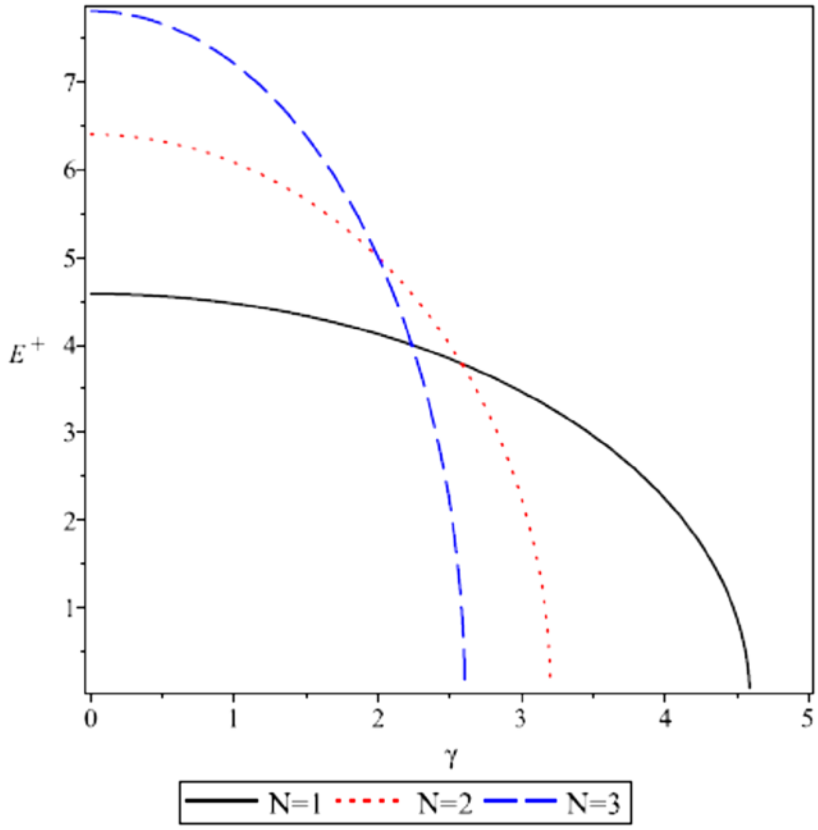

FIG. 7: (E^+ vs. γ) for $\varepsilon = 0$ (Dirac Oscillator).

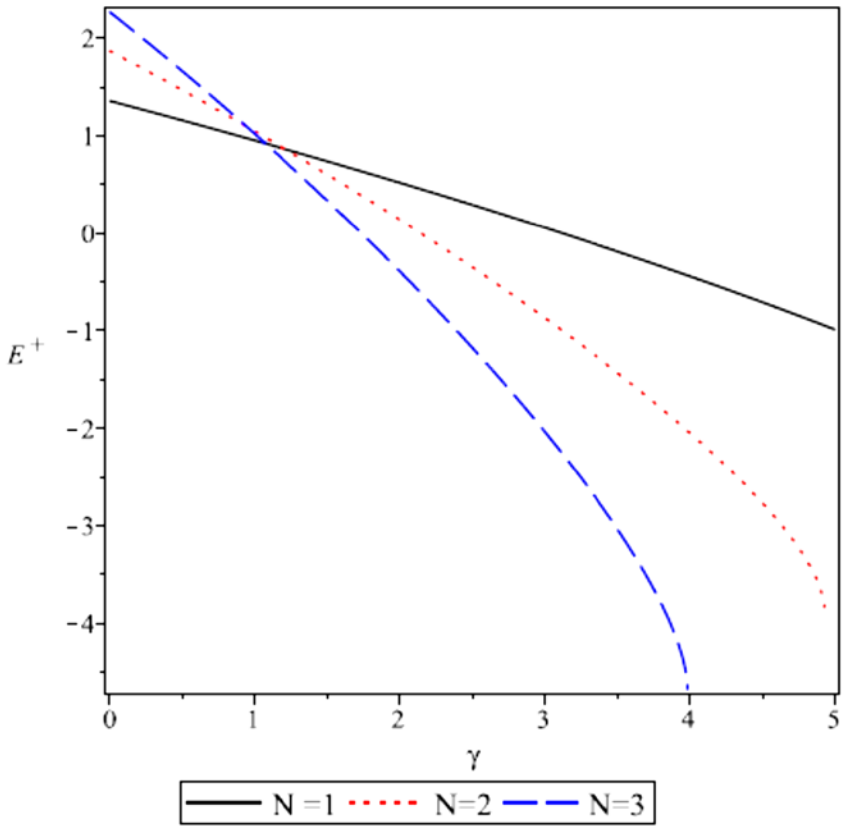

FIG. 8: (E^+ vs. γ) for $\varepsilon = 9$ (Dirac Oscillator).

Some special cases :

We will consider some special cases for vanishing γ and ε.

For $\gamma = 0$, namely in the absence of deformation, we replace $\gamma = 0$ in (2.86) find,

$$E^{\pm} = \pm\Omega\sqrt{m^2 + 2m\omega\Omega N}. \tag{2.88}$$

The case for $\varepsilon = 0$, namely in the absence of an electric field implies $\Omega = 1$, and the expression of the energy spectrum (2.86) becomes

$$E^{\pm} = \pm\sqrt{m^2 + 2m\omega N - \gamma^2 N^2}. \tag{2.89}$$

In the case where $\gamma = \varepsilon = 0$, we have the pure Dirac oscillator case. This limit yields

$$E = \pm\sqrt{m^2 + 2Nm\omega}, \tag{2.90}$$

which is in agreement with the result of the ordinary case.

In this contribution, we studied the exact solutions of one-dimensional Klein-Gordon and Dirac oscillators subject to a uniform electric field in the context of the new type of the extended uncertainty principle using the displacement operator method. The energy eigenvalues and eigenfunctions are determined for both cases. In the Klein-Gordon oscillator case, the wave functions are expressed in terms of the associated Laguerre polynomials and in the Dirac oscillator case, the wave functions are obtained in terms of the confluent Heun functions. In the latter case, the energy eigenvalues are obtained by the polynomial reduction of the confluent Heun functions. The analytical expression of the energy spectrum contains corrections of all orders of $(\varepsilon\gamma)^2$. This is related to the exact contribution to the Stark effect in this deformation framework and it varies with the power of n^2, which explains the confinement phenomenon. For large values of n, the square of the energy spectrum $(E)^2$ becomes negative and, in order to ensure positivity of the square of the energy, one must impose an upper bound on the allowed values of n.

The energy eigenvalues are plotted as a function of n for various numerical values of the parameter γ in order to show our result graphically. The limiting cases are also studied using the special values of the physical parameters for both the Klein-Gordon and Dirac oscillator. It is remarkable that the results obtained in this context of the displacement operator can be interpreted as the case of systems with variable masses depending on the position. This study really needs more details, will make the goal of a future project.

Chapitre 3

Treatment of spinless particle on the de Sitter and the Anti-de Sitter spaces :

3.1 Introduction

In these research works [36, 37] , Mignemi showed that it can be derived from the definition of quantum mechanics on a de Sitter background with a suitably chosen parametrization, that is, the Heisenberg uncertainty principle should be modified in a (anti)- de Sitter background by introducing corrections proportional to the cosmological constant $\wedge = \frac{3}{R^2}$, where $R^2 \prec 0$ for de Sitter space-time, and $R^2 \succ 0$ for anti-de Sitter space-time[99]. This modification known in the literature by the extended uncertainty principle (EUP), it can be achieved by modifying the usual canonical commutation relations. Over past decades, the implications of this (EUP) hypothesis have developed significantly and many works are examined for quantum mechanics and classical on the background (anti) -de Sitter[38, 39, 44, 46].

In this analysis, first we are interested to study two fundamental problems of quantum mechanics in the context of (anti)-de Sitter spaces :

-To establish the exact solutions of the (1+3)-dimensional Klein-Gordon oscillator.

-To determine the corrections to the spectrum of the Klein-Gordon equation for the coulomb plus scalar potentials using the perturbation theory. This gives rise to the appearance of a minimal uncertainty in momentum. On the other, we also study the effect of the deformation and the changes made to relativistic system in the framework of the extended uncertainty principle.

3.2 Review of the deformed quantum mechanics relation : de Sitter and anti-de Sitter spaces

The extended uncertainty principle (EUP) can be obtained from the definition of quantum mechanics on (anti) de sitter (dS) space-time. It is well known that (anti)-de Sitter space-time can be realized as a hyperboloid of equation $\eta_{ab}\zeta^a\zeta^b = \pm R^2$ embedded in five-dimensional Minkowski space with coordinates $\zeta^a(a = 0, 1, 2, 3, 4)$ and metric $\eta_{ab} = diag\,(1, -1, -1, -1, \pm 1)$, when $R \rightarrow \infty$ the de Sitter (dS) invariant special relativity (SR) will be reduced to ordinary special relativity [100]

$$ds^2 = \eta_{ab} d\zeta^a d\zeta^b = B_{\mu\nu}(x)\, dx^\mu dx^\nu; \quad \mu = \nu = 0; 1; 2; 3, \tag{3.1}$$

where the parametrization of the hyperboloid is given by projective (Beltrami) coordinates [101, 102],

$$x_\mu = \frac{\zeta_\mu}{\zeta_4} \tag{3.2}$$

and

$$B^{\mu\nu}(x) = \left(1 - \frac{\eta_{\sigma\tau}x^\sigma x^\tau}{R^2}\right)\left(\eta^{\mu\nu} - \frac{x^\mu x^\nu}{R^2}\right), \tag{3.3}$$

is Beltrami metric. Note that, the Beltrami coordinate system, is similar to the Minkowski one in a flat space-time, and the Beltrami de sitter ($\mathcal{B}dS$) space-time is the dS space-time with Beltrami metric. The generators of de Sitter in Beltrami coordinates and the momentum operators satisfy the following commutation relations [36, 37][103, 104, 105]

$$[J_{\mu\nu}, J_{\sigma\rho}] = i\left(\eta_{\nu\rho}J_{\mu\sigma} - \eta_{\nu\sigma}J_{\mu\rho} + \eta_{\mu\sigma}J_{\nu\rho} - \eta_{\mu\rho}J_{\nu\sigma}\right), \tag{3.4}$$

$$[J_{\mu\nu}, p_\rho] = i\left(\eta_{\mu\rho}p_\nu - \eta_{\nu\rho}p_\mu\right); \qquad [p_\mu, p_\nu] = \frac{iJ_{\mu\nu}}{R^2}, \tag{3.5}$$

and

$$[x_\mu, p_\nu] = i\left(\eta_{\mu\nu} + \frac{x_\mu x_\nu}{R^2}\right); \qquad [x_\mu, x_\nu] = 0. \tag{3.6}$$

where $\mu\ \nu = 0{,}1, 2, 3$ and $J_{\mu\nu}$ are the generators of Lorentz transformations given by $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$

In the theory of SR on (A)dS space-time there are two universal parameters : the speed of light c and the cosmological constant $\wedge$ [100].

The non-relativistic modified commutation relations leading to the extended commutation relations, is given by [39]

$$\begin{cases} [X_j, P_k] = i\hbar\left(\delta_{jk} + \alpha X_j X_k\right), \\ [X_j, X_k] = 0, \\ [P_j, P_k] = i\hbar\alpha L_{jk}, \end{cases} \tag{3.7}$$

where $j, k = 1, 2, 3$, $L_{jk} = X_j P_k - X_k P_j$, and α being the constant deformation parameter, where α is a positive parameter proportional to the cosmological constant or inversely proportional to the square of the anti-de Sitter radius ($\alpha = H^2$:H^2 is the Hubble rate)[41], and in the limit $\alpha \longrightarrow 0$, we recover the canonical commutation relations from standard quantum mechanics.

As the case of ordinary quantum mechanics, the commutation relation (3.7) lead to the following extended uncertainty principle (EUP)

$$(\Delta X_i)(\Delta P_i) \succeq \frac{\hbar}{2}\left(1 + \alpha\left(\Delta X_i\right)^2\right), \tag{3.8}$$

which implies the appearance of a nonzero minimal uncertainty in momentum. The minimization of (3.8) with respect to ΔX_i gives

$$(\Delta P_i)_{\min} = \hbar\sqrt{\alpha}, \qquad \forall \quad k. \tag{3.9}$$

The most representation of the position and momentum operators obeying relation (3.7) is given by

$$X_i = x_i; \qquad P_i = \frac{\hbar}{i}\left(\delta_{ij} + \alpha x_i x_j\right)\frac{\partial}{\partial x_i}, \tag{3.10}$$

where the operators x_i and p_j satisfy the canonical commutation relation $[x_i, p_j] = i\hbar\delta_{ij}$. Using the symmetrically condition of the operators of position and momentum, the modified scalar product can be written as

$$\langle\phi\,|\psi\rangle = \int \frac{d^3\mathbf{r}}{(1+\alpha r^2)^2}\phi^{\times}(\mathbf{r})\,\psi(\mathbf{r})\,; \qquad \text{where } r = \sum_{i=1}^{3} x_i^2. \tag{3.11}$$

Now, the extended uncertainty principle for the de Sitter space dS space can be constructed by replacing $\alpha \longrightarrow -\alpha$, in this case and contrary to the previous case, we will have,

$$(\Delta X_i)(\Delta P_i) \geq \frac{\hbar}{2}\left(1 - \alpha\,(\Delta X_i)^2\right). \tag{3.12}$$

let's notice this relation does not give the minimal uncertainty in momentum, we get

$$-\frac{(\Delta P_i)}{\alpha\hbar} - \frac{1}{\alpha}\sqrt{\alpha + \frac{(\Delta P_i)^2}{\hbar^2}} \leq (\Delta X_i) \leq -\frac{(\Delta P_i)}{\alpha\hbar} + \frac{1}{\alpha}\sqrt{\alpha + \frac{(\Delta P_i)^2}{\hbar^2}}. \tag{3.13}$$

and in the limit $(\Delta P_i) \rightarrow 0$ the space become finite $-\frac{1}{\sqrt{\alpha}} \leq (\Delta X_i) \leq \frac{1}{\sqrt{\alpha}}$.

A representation of X_i and P_i that satisfies for the de Sitter space dS space, may be taken as

$$X_i = x_i; \quad P_i = \frac{\hbar}{i}\left(\delta_{ij} - \alpha x_i x_j\right)\frac{\partial}{\partial x_j}. \tag{3.14}$$

In the following section, we examine the Klein Gordon oscillator and The Klein-Gordon equation with a Coulomb plus scalar potential in anti-de Sitter space. we put $(\hbar = c = 1)$

3.3 The (1+3)-dimensional Klein Gordon oscillator in AdS space

In this section, we are interested in solving the (1+3)-dimensional Klein Gordon oscillator, in position space with deformed commutation relations. In this case, the stationary equation describing the Klein Gordon oscillator in (1+3)-dimension is given by

$$\left[\left(E^2 - m^2\right) - \left(\mathbf{P} + im\omega\mathbf{r}\right)\left(\mathbf{P} - im\omega\mathbf{r}\right)\right]\Phi\left(\mathbf{r}\right) = 0, \tag{3.15}$$

where m is the rest mass, and ω is the classical frequency of the oscillator.

Applying the definition of the position and momentum operators reported in sect (2), the momentum squared operator can be expressed as

$$P^2 = -\left[\left(1+\alpha r^2\right)\frac{\partial}{\partial r}\right]^2 - \frac{2}{r}\left(1+\alpha r^2\right)\frac{\partial}{\partial r} + \frac{L^2}{r^2}, \tag{3.16}$$

and the Klein Gordon oscillator Eq (3.15) can be rewritten as the following differential equation :

$$\left(m^2 - E^2\right)\Phi = \left\{\left[\left(1+\alpha r^2\right)\frac{\partial}{\partial r}\right]^2 + \frac{2}{r}\left(1+\alpha r^2\right)\frac{\partial}{\partial r} - \frac{\hat{L}^2}{r^2} - m^2\omega^2 r^2 + m\omega\left(3+\alpha r^2\right)\right\}\Phi\left(\mathbf{r}\right). \tag{3.17}$$

Thus, it's appropriate to split the energy eigenfunction Φ into a radial part and an angular part as :

$$\Phi\left(\mathbf{r}\right) = R_{n,\ell}\left(r\right) Y_{\ell,m}\left(\theta,\varphi\right), \tag{3.18}$$

where $Y_{n,\ell}$ are the eigenfunction of the angular part.

$$\hat{L}^2 Y_{\ell,m}(\theta,\varphi) = \ell(\ell+1) Y_{\ell,m}(\theta,\varphi) \tag{3.19}$$

This allows us to rewrite Eq. (3.17) as

$$\left[\left[\left(1+\alpha r^2\right)\frac{d}{dr}\right]^2 + \frac{2}{r}\left(1+\alpha r^2\right)\frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} - m^2\omega^2 r^2 + m\omega\alpha r^2 + E^2 - m^2 + 3m\omega\right] \mathrm{R}_{n,\ell}(r) = 0. \tag{3.20}$$

To solve this equation, we begin by making the following change of variable

$$\sqrt{\alpha}\rho = \tan^{-1}\sqrt{\alpha} r, \tag{3.21}$$

which maps the interval $r \in \left]0,\infty\right[$ to $\rho \in \left]0, \frac{\pi}{2\sqrt{\alpha}}\right[$ and brings Eq. (3.20) to the following form

$$\left[\frac{d^2}{d\rho^2} + \frac{2\sqrt{\alpha}}{\tan\left(\sqrt{\alpha}\rho\right)}\frac{d}{d\rho} - \frac{\alpha\ell(\ell+1)}{\tan^2\left(\sqrt{\alpha}\rho\right)} - m\omega\left(\frac{m\omega}{\alpha} - 1\right)\tan^2\left(\sqrt{\alpha}\rho\right) + E^2 - m^2 + 3m\omega\right] \mathrm{R}_{n,\ell}(\rho) = 0. \tag{3.22}$$

To eliminate the first derivative, we introduce the following ansatz

$$R_{n,\ell}(\rho) = e^{-\sqrt{\alpha}\int^{\rho}\frac{d\zeta}{\tan\left(\sqrt{\alpha}\zeta\right)}} g_{n,\ell}(\rho), \tag{3.23}$$

after some manipulation, we obtain

$$\left[\frac{d^2}{d\rho^2} - \frac{\alpha\ell(\ell+1)}{\tan^2\left(\sqrt{\alpha}\rho\right)} - m\omega\left(\frac{m\omega}{\alpha} - 1\right)\tan^2\left(\sqrt{\alpha}\rho\right) + E^2 - m^2 + 3m\omega + \alpha\right] g_{n,\ell}(\rho) = 0. \tag{3.24}$$

Introducing now the following change of function

$$g_{n,\ell}(\rho) = \sin^{\ell+1}\left(\sqrt{\alpha}\rho\right)\cos^{\sigma}\left(\sqrt{\alpha}\rho\right) F_{n,\ell}(\rho), \tag{3.25}$$

where σ is a constant to be determined letter. By means of the substitution given in Eq.

(3.25), the last differential equation (3.24)take the following form :

$$\left[\begin{array}{c}\frac{d^2}{d\rho^2}+2\sqrt{\alpha}\left(\frac{(\ell+1)}{\tan\left(\sqrt{\alpha}\rho\right)}-\sigma\tan\left(\sqrt{\alpha}\rho\right)\right)\frac{d}{d\rho}-\alpha\sigma\left(2\ell+3\right)\\ +\alpha\left[\sigma\left(\sigma-1\right)-\frac{m\omega}{\alpha}\left(\frac{m\omega}{\alpha}-1\right)\right]\tan^2\left(\sqrt{\alpha}\rho\right)+E^2-m^2+3m\omega-\ell\alpha\end{array}\right]F_{n,\ell}\left(\rho\right)=0. \tag{3.26}$$

To eliminate the term $\tan^2\left(\sqrt{\alpha}\rho\right)$ by demanding

$$\sigma\left(\sigma-1\right)-\frac{m\omega}{\alpha}\left(\frac{m\omega}{\alpha}-1\right)=0, \tag{3.27}$$

then it leads to the following expression of σ

$$\sigma_+=\frac{m\omega}{\alpha},\sigma_-=1-\frac{m\omega}{\alpha}. \tag{3.28}$$

Among these two solutions, the physically acceptable one is only σ_+, the second solution leads to a non physically acceptable wave function. Then Eq. (3.26) simplifies to

$$\left[\frac{d^2}{d\rho^2}+2\sqrt{\alpha}\left(\frac{\ell+1}{\tan\left(\sqrt{\alpha}\rho\right)}-\frac{m\omega}{\alpha}\tan\left(\sqrt{\alpha}\rho\right)\right)\frac{d}{d\rho}-2\ell m\omega+E^2-m^2-\alpha\ell\right]\mathrm{F}_{n,\ell}\left(\rho\right)=0. \tag{3.29}$$

At this stage, we introduce another change of variable defined by

$$\eta=2\sin^2\left(\sqrt{\alpha}\rho\right)-1.\ \text{ with }\quad -1\preceq\eta\preceq 1 \tag{3.30}$$

the equation (3.29) reduces to

$$\left[\left(1-\eta^2\right)\frac{d^2}{d\eta^2}+\left(\ell-\frac{m\omega}{\alpha}+1-\left(\ell+\frac{m\omega}{\alpha}+2\right)\eta\right)\frac{d}{d\eta}+\frac{E^2-m^2-\alpha\ell-2\ell m\omega}{4\alpha}\right]\mathrm{F}_{n,\ell}\left(\eta\right)=0. \tag{3.31}$$

which is exactly the Jacobi polynomials differential equation $P_n^{(a,b)}\left(\eta\right)$ whose parameters a and b are given by imposing the following constraint

$$\frac{E^2-m^2-\alpha\ell-2\ell m\omega}{4\alpha}=n\left(n+a+b+1\right), \tag{3.32}$$

$$a=\frac{m\omega}{\alpha}-\frac{1}{2};\quad b=\ell+\frac{1}{2}, \tag{3.33}$$

where n is non-negative integer and the solution can be written in terms of Jacobi polynomials as

$$F_{n,\ell}(\eta)=P_n^{\left(\frac{m\omega}{\alpha}-\frac{1}{2},\ell+\frac{1}{2}\right)}(\eta). \tag{3.34}$$

Using the the former variable r, we will have the following final form of the wave function Φ :

$$\Phi_{n,\ell}(\mathbf{r})=\mathcal{C}\frac{r^\ell}{(1+\alpha r^2)^{\frac{m\omega}{2\alpha}+\frac{\ell}{2}}}P_n^{\left(\frac{m\omega}{\alpha}-\frac{1}{2},\ell+\frac{1}{2}\right)}\left(\frac{\alpha r^2-1}{1+\alpha r^2}\right)Y_{\ell,m}(\theta,\varphi), \tag{3.35}$$

where $\mathcal{C}$ is the normalization constant.

To determine the expressions of the energy spectrum of Klein Gordon oscillator , using the condition (3.32) and replacing the parameters a,and b by their expressions (3.33), we finally get the following result

$$E_{n,l}^{\pm AdS}=\pm\sqrt{m^2+2m\omega(2n+\ell)+\alpha[4n(n+l+1)+l]}, \tag{3.36}$$

where $\pm$ denotes the positive (negative) energy solutions associated respectively with the particle and the antiparticle for relativistic quantum systems.

Notice that the energy levels depend on the quantum number n and n^2 and for large n it is asymptotic to

$$E_n^{\pm AdS}\rightarrow\pm 2\sqrt{\alpha}n. \tag{3.37}$$

This effect is due to the modification of the Heisenberg algebra. As a result, we remark that for a fixed value of n, the energy $E_{n,l}^{+AdS}$ increases monotonically with the increase of the EUP parameter α. Expanding the expression of the energy levels to first order in α, we obtain

$$E_{n,l}^{\pm AdS}=\pm\sqrt{m^2+2m\omega(2n+\ell)}\left(1+\frac{\alpha}{2}\frac{(4n(n+l+1)+l)}{(m^2+2m\omega(2n+\ell))}\right). \tag{3.38}$$

The first term is the energy spectrum of the ordinary 3d Klein-Gordon oscillator, while the second term is the corrections brought about by the existence of nonzero minimal uncertainty in momentum, and when we study the limit $\alpha \rightarrow 0$, we obtain

$$E_{n,l}^{\pm AdS} = \pm\sqrt{m^2 + 2m\omega\,(2n+\ell)}, \tag{3.39}$$

which is the same result in ordinary case. Before finishing this section, let us see the influence of the EUP in dS on the energy eigenvalues ($\alpha \rightarrow -\alpha$) and by the same steps and same techniques, we arrive

$$E_{N,j}^{\pm dS} = \pm\sqrt{m^2 + 2m\omega\,(2n+\ell) - \alpha\,[4n\,(n+l+1)+l]}. \tag{3.40}$$

In this case, for large values of n , the square of the energy spectrum $\left(E_{n,\mu}^{dS}\right)^2$ becomes negative. In order to ensure positivity of the the square of the energy, one must impose an upper bound on the allowed values n and l.

3.4 The Klein-Gordon equation with a Coulomb plus scalar potential in AdS space

The hydrogen atom is a fundamental problem of quantum mechanics ; it is of considerable importance in atomic and molecular physics. it allows to understand the spectra of hydrogenoids and to explain the structure of the energy levels and the spectra of the atoms in the case of models with independent electrons or approach of an average field. Furthermore, the hydrogen atom has grown enormously ;especially in the context of deformed algebras and several papers have been studied. In non relativistic case, the spectrum and eigenfunctions in the momentum representation for 1D Coulomb-like potential with deformed Heisenberg algebra are found exactly in [106, 107], for higher dimensions, the problem becomes complicated, only perturbative solutions have been found

[[108] − [111]]. On the contrary, in the case of relativistic quantum mechanics, no study is presented, accordingly, our attempt through this letter will be addressed the problem in question for the case of the Klein Gordon equation in the framework of anti-de Sitter spaces.

To study the eigenvalue problem for hydrogen atom in 3-dimensional case we start considering a standard Hamiltonian :

$$\left\{P^2 + (M + V_s(r))^2 - (E + V_v(r))^2\right\} \psi(\mathbf{r}) = 0 \ , \tag{3.41}$$

where M and E denote the mass and the energy of the particle, respectively and $r = \sqrt{\sum_{j=1}^{3} X_j^2}$ and $P = \sqrt{\sum_{j=1}^{3} P_j^2}$ satisfy deformed commutation relation (3.7). The Coulomb potential and the scalar potential are taken as

$$V_s(r) = -\frac{V_s}{r} \quad V_v(r) = -\frac{V_v}{r}. \tag{3.42}$$

The scalar potential is added to the mass term in the Klein Gordon equation and may be understood as an effective position-dependent mass, which is of considerable significance in various areas of physics, citing for instance quantum well and quantum dots[112, 113], in the description of electronic properties and band structure of semiconductor heterostructures [114], ...etc.

Now, we apply the definition for X_j and P_j (3.10) reported in section 2, the momentum squared operator (3.16)can be expressed

$$P^2 = -\left(1 + \alpha r^2\right)^2 \frac{\partial^2}{\partial r^2} - \left(1 + \alpha r^2\right) 2\alpha r \frac{\partial}{\partial r} - \frac{2}{r}\left(1 + \alpha r^2\right) \frac{\partial}{\partial r} + \frac{L^2}{r^2}, \tag{3.43}$$

if we expand (3.43) at the first order in α, we have

$$P^2 = -\left(1 + 2\alpha r^2\right) \frac{\partial^2}{\partial r^2} - \frac{2}{r}\left(1 + 2\alpha r^2\right) \frac{\partial}{\partial r} + \frac{L^2}{r^2} + \mathcal{O}\left(\alpha^2\right), \tag{3.44}$$

therefore, the Klein Gordon Eq (3.41) can be written as follows

$$\left\{-\left(1+2\alpha r^2\right)\frac{\partial^2}{\partial r^2}-\frac{2}{r}\left(1+2\alpha r^2\right)\frac{\partial}{\partial r}+\frac{L^2}{r^2}+(M-\frac{V_s}{r})^2-(E+\frac{V_v}{r})^2\right\}\psi(\mathbf{r})=0, \tag{3.45}$$

or as follows;

$$\left\{-\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)-2\alpha r^2\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)+\frac{L^2}{r^2}+(M-\frac{V_s}{r})^2-(E+\frac{V_v}{r})^2\right\}\psi(\mathbf{r})=0, \tag{3.46}$$

and using this replacement

$$\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}=\frac{1}{r}\frac{\partial^2}{\partial r^2}r, \tag{3.47}$$

the Eq (3.46) becomes

$$\left\{-\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r\right)-2\alpha r^2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r\right)+\frac{L^2}{r^2}+(M-\frac{V_s}{r})^2-(E+\frac{V_v}{r})^2\right\}\psi(\mathbf{r})=0. \tag{3.48}$$

To solve this equation, using this separate form;

$$\psi(\mathbf{r})=\frac{R^{\alpha}(r)}{r}Y_{l,m}\left(\theta,\varphi\right). \tag{3.49}$$

where $Y_{l,m}\left(\theta,\varphi\right)$ are spherical harmonics, eigenvectors of the orbital kinetic moment

$$L^2Y_{l,m}\left(\theta,\varphi\right)=\ell\left(\ell+1\right)Y_{l,m}\left(\theta,\varphi\right)=\left(k^2-\frac{1}{4}\right)Y_{l,m}\left(\theta,\varphi\right), \tag{3.50}$$

with $k=l+\frac{1}{2}$. Substitution (3.49) and (3.50) into Eq. (3.48), We obtain the radial equation of the Klein-Gordon equation in AdS space :

$$\left[-\frac{d^2}{dr^2}+\frac{k^2+V_s^2-V_v^2-\frac{1}{4}}{r^2}+\left(M^2-E^2\right)-\frac{2\left(MV_s+EV_v\right)}{r}-2\alpha r^2\left(\frac{d^2}{dr^2}\right)+\mathcal{O}\left(\alpha^2\right)\right]\mathrm{R}^{\alpha}(\mathrm{R})=0, \tag{3.51}$$

which can be written as

$$\left[H_0 + \alpha W + \mathcal{O}\left(\alpha^2\right)\right] R^{\alpha}(r) = 0, \tag{3.52}$$

with H_0 represents the undisturbed Hamiltonian corresponds to the ordinary case $\alpha = 0$ of the Klein-Gordon equation for Hydrogen atom with scalar potential given by

$$H_0 = -\frac{d^2}{dr^2} + \frac{k^2 + V_s^2 - V_v^2 - \frac{1}{4}}{r^2} + \left(M^2 - E^2\right) - \frac{2\left(MV_s + EV_v\right)}{r} \tag{3.53}$$

and W is the disturbed Hamiltonian

$$W = -2r^2 \frac{d^2}{dr^2}. \tag{3.54}$$

To simplify the shape of H_0 and W , introducing this notation

$$\beta = \frac{EV_v + MV_s}{\sqrt{M^2 - E^2}}\ ,\ \nu = \sqrt{k^2 + V_s^2 - V_v^2} - \frac{1}{2} \text{ and } a = \sqrt{M^2 - E^2} \quad , \tag{3.55}$$

we will then have

$$H_0 = -\frac{d^2}{dr^2} + \frac{\nu\left(\nu + 1\right)}{r^2} + a^2 - \frac{2a\beta}{r} \tag{3.56}$$

and the new the expression form for W,

$$W = -2\nu\left(\nu + 1\right) + 2r^2\left(H_0 - a^2\right) + 4ra\beta. \tag{3.57}$$

We have used (3.53).

In order to study the influence of this deformation on the energy levels of the hydrogen atom we will consider the term αW as perturbation in ordinary quantum mechanics. Therefore, the perturbation theory can be used to calculate the correction to the energy levels of the hydrogen atom in the first-order in α and to avoid complex spectra, subsequently, we consider the case of the weak Coulomb potential such that $k^2+V_s^2-V_v^2>0$, otherwise the solution becomes oscillatory .

Now, for $\alpha=0$, the exact solution of the ordinary Klein Gordon equation for Hydrogen atom can be found in [115, 116]. The eigenvalues and the corresponding normalized eigenfunctions expressed according to Laguerre's polynomial are given by

$$R^0_{n'\nu}(r)=N_{n'l}\frac{n'!\Gamma(2\nu+2)}{\Gamma(2\nu+2+n')}\left(2ar\right)^{\nu+1}e^{-ar}L^{2\nu+1}_{n'}(2ar) \tag{3.58}$$

where $N_{n'l}$ is normalization constant determined by this condition

$$\int R^{0*}_{n'\nu}(r)R^0_{n'\nu}(r)dr=1. \tag{3.59}$$

By using the recursion relation for Laguerre polynomials [117]

$$xL^{2\nu+1}_n=2(n+\nu+1)L^{2\nu+1}_n-(n+1)L^{2\nu+1}_{n+1}-(n+2\nu+1)L^{2\nu+1}_{n-1}. \tag{3.60}$$

and

$$d_n^2=\int x^{\alpha}e^{-x}L_n^{\alpha}\left(x\right)L_n^{\alpha}\left(x\right)dx=\frac{\Gamma(n+\alpha+1)}{n!} \tag{3.61}$$

where d_n^2 is the square of the norm of $L_n^{\alpha}\left(x\right)$, the normalized radial functions are

$$R^0_{n'\nu}(r)=\sqrt{\frac{an'!}{\left(\nu+n'+1\right)\Gamma(2\nu+2+n')}}\left(2ar\right)^{\nu+1}e^{-ar}L^{2\nu+1}_{n'}(2ar) \tag{3.62}$$

and the corresponding energy spectrum, eigenvalues of the radial part of the Klein-Gordon equation with a Coulomb potential and scalar potential is deducted by this condition $\beta-\nu-1=n'$.

$$\mathrm{E}^{\alpha=0\pm}_{n,\ell}=\mathrm{M}\left\{-\frac{V_sV_v}{V_v^2+\beta^2}\pm\left[\left(\frac{V_sV_v}{V_v^2+\beta^2}\right)^2-\frac{V_s^2-\beta^2}{V_v^2+\beta^2}\right]^{\frac{1}{2}}\right\}, \tag{3.63}$$

or

$$\mathrm{E}_{n,\ell}^{\alpha=0\pm}=\mathrm{M}\left\{-\frac{V_sV_v}{V_v^2+(\nu+n-l)^2}\pm\left[\left(\frac{V_sV_v}{V_v^2+(\nu+n-l)^2}\right)^2-\frac{V_s^2-(\nu+n-l)^2}{V_v^2+(\nu+n-l)^2}\right]^{\frac{1}{2}}\right\}, \tag{3.64}$$

where we have introduced the principal quantum number : $n=n'+l+1$.

Now, to determine the correction of the energy levels associated with the disturbed Hamiltonian W (3.57) due to the anti-de sitter space-time, we use the first-order perturbation theory in the deformation parameter α,

$$\alpha E_n^{(1)}=\alpha\int R_{nl}^{0*}(r)\,(W)\,R_{nl}^{0}(r)dr$$

$$=\alpha\left[-2\nu\,(\nu+1)<r^{(0)}>+4a\beta<r^{(1)}>-2a^2<r^{(2)}>\right] \tag{3.65}$$

where

$$<r^{(m)}>\int r^m R_{nl}^{0*}(r)R_{nl}^{0}(r)dr. \tag{3.66}$$

For the calculation of expectation of value of $<r^{(m)}>$, we take advantage of the properties (3.60) and (3.61) and a straightforward and long calculation leads to

$$\begin{cases} <r^{(0)}>=1 \\ <r^{(1)}>=\int rR_{nl}^{0*}(r)R_{nl}^{0}(r)dr=\frac{1}{2a(\nu+n-l)}\left[3(\nu+n-l)^2-\nu\,(\nu+1)\right] \\ <r^{(2)}>=\int r^2R_{nl}^{0*}(r)R_{nl}^{0}(r)dr=\frac{1}{2a^2}\left[5(\nu+n-l)^2+1-3\nu\,(\nu+1)\right] \end{cases} \tag{3.67}$$

Then the first order of the perturbation theory takes this form

$$\alpha E_n^{(1)}=\alpha\left[(\nu+n-l)^2-\nu\,(\nu+1)-1\right]$$

$$=\alpha\left\{\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}\right)\left(\sqrt{k^2+V_s^2-V_v^2}+\frac{1}{2}\right)-1\right\} \tag{3.68}$$

which represents the quantum fluctuations due to the extended uncertainty principle on (anti) -de sitter space-time, depending on the powers in n^2 , explains the phenomenon of confinement and the expression of the hydrogen atom energy levels is modified as

$$
{}^{AdS}\mathrm{E}_{n,\ell}^{\alpha\pm}=\mathrm{M}\left\{-\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right.
$$

$$
\left.\pm\left[\left(\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right)^2-\frac{V_s^2-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right]^{\frac{1}{2}}\right\},
$$

$$
+\alpha\left\{\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}\right)\left(\sqrt{k^2+V_s^2-V_v^2}+\frac{1}{2}\right)-1\right\}+O\left(\alpha^2\right). \tag{3.69}
$$

In this last expression of the spectrum (3.69), we notice that the spectrum energy on anti-de Sitter is bigger than the energy in ordinary case. Before concluding this paragraph, we would like to see the influence of space dS on the eigenvalues of the system ($\alpha \to -\alpha$). By the same steps, the energy eigenvalues of the system will have the following form

$$
{}^{ds}\mathrm{E}_{n,\ell}^{\alpha\pm}=\mathrm{M}\left\{-\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right.
$$

$$
\left.\pm\left[\left(\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right)^2-\frac{V_s^2-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right]^{\frac{1}{2}}\right\},
$$

$$
-\alpha\left\{\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}\right)\left(\sqrt{k^2+V_s^2-V_v^2}+\frac{1}{2}\right)-1\right\}+O\left(\alpha^2\right). \tag{3.70}
$$

In this case, we can see that the energy spectrum (3.70) on the de Sitter space is smaller than the energy in ordinary case.

In this contribution, we have investigated the three-dimensional Klein-Gordon oscillator and the Klein-Gordon equation with a Coulomb plus scalar potential in the context of quantum deformations for the (anti)- de Sitter algebras. For the 3-dimensionals Klein-Gordon oscillator, according to the symmetry of the system, we used the adequate radial representation and some change of variables, the problem has been converted to a differential equation of type Jacobi polynomials. The energy eigenvalues and their corresponding eigenfunctions are exactly and analytically obtained . For the case of the Klein-Gordon equation for hydrogen atom, the problem is complicated and in order to determine energy spectra, the perturbation theory has been applied to calculate the correction to the energy levels in the first-order in α. In both problem, we show that the spectrum energy on anti-de Sitter is bigger than the energy in ordinary case contrariwise in ds space , the energy spectrum is smaller than the energy in ordinary case.

Chapitre 4

Reformulation of supersymmetric Feynman's approach in the context of deformed algebras :the EUP Dirac Oscillator

4.1 Introduction

It is well known that the spin is a fundamental physical quantity in quantum physics and plays a significant role in various areas of physics, in particular, in the explanation of the mesoscopic phenomena. In the relativistic case, the exact analytical solutions of physical models much required, enables us to explore at the same time relativistic and spin effects, and the relativistic principles require that space-time must be described in the unified manner. Indeed, Feynman's path integral formulation for systems with spin has not yet been definitively achieved due to the discrete nature of the spin and the requirements of relativistic invariance. In fact, path integral uses classical and continuous concepts such as trajectories whereas the spin is irreducibly of a discrete nature, without classical equivalent, and to satisfy the relativistic invariance requirements on the other hand. To overcome this difficulty within this framework, some models were presented for this purpose. For example, the Feynman attempt for the free Dirac electron using the Poisson stochastic process [118], the Schulman description of the spin of a nonrelativistic particle by the top model using the three Euler angles [119], and its extension to the relativistic case [120], the Barut–Zanghi theory for the classical spinning electron related to zitterbewegung [121], the Bosonic and Fermionic Schwinger model in the related coherent state space [122, 123, 124] and the supersymmetric model using the Grassmann variables for the spin evolution with many developments [125, 126, 127, 128].

Recently, the applicability of this Feynman formulation for the spin system has undergone notable development in various domains of physics with different topologies modeled by deformed algebras. For example, if you take the effects of the gravitational field in quantum mechanics in presence of the generalized uncertainty principle (GUP), a significant number of papers have been published. Citing for instance, the spinning particle subjected to the action of combined vector and scalar potentials [31, 68] and the Dirac oscillator [69, 70]. In noncommutative space, the Klein-Gordon and Dirac oscillators [72], and the harmonic oscillator related to energy-dependent potential [73]. And others important similar references using path approach as, the Klein-Gordon equation with the energy dependent linear and Coulomb potentials is treated in [78] and the harmonic oscillator and the radial hydrogen atom propagators related to energy-dependent potentials are analyzed [77].

Our attempt through this work is to set up a path integral formulation to establish the Green function for the Dirac oscillator problem in the context of another type of deformation due to the topology of the physical space called the extended uncertainty principle (EUP) , as an example in the (anti)-de sitter background, the Heisenberg uncertainty principle is modified by introducing corrections proportional to the cosmological constant.

At the same time, it is important to remember that the Dirac oscillator was introduced for the first times by [129, 130]. It was the subject of many developments and received considerable study in various areas of physics. For example, it appears in quantum optics [131], in nuclear physics [132], in noncommutative geometry [133] and in graphene physics [134]. It is used as the confining part of the phenomenological Cornell potential and an intergroup potential in quantum chromodynamics.

4.2 Brief review of (anti)-de Sitter one-dimensional background

In this section, before starting the calculations, it is important to expose some useful formulas and notions which will be used later. In this case of one dimensions, the expressions of the following relations (3.7), (3.8), (3.10) ,(3.12) and (3.14) become :

- For the Anti-de Sitter space,the modified commutation relations leading to the extended commutation relations is given as [136],

$$\left[\hat{X},\hat{P}\right]=i\left(1+\alpha\hat{X}^{2}\right), \tag{4.1}$$

where α is a positive deformation parameter proportional to the cosmological constant, or inversely proportional to the square of the anti-de Sitter radius ($\alpha = H^2$: H^2 is the Hubble rate). Which lead to the following EUP :

$$\left(\Delta X\right)\left(\Delta P\right)\geqslant\frac{1}{2}\left(1+\alpha\left(\Delta X\right)^{2}\right), \tag{4.2}$$

According to (4.1), the $\hat{X}$ and $\hat{P}$ operators in this representation can be realized by operators $\hat{x}$ and $\hat{p}$, as follows :

$$\begin{cases} \hat{X}=\hat{x} \\ \hat{P}=\left(1+\alpha x^{2}\right)\hat{p} \end{cases}, \tag{4.3}$$

with $\hat{x}$ and $\hat{p}$ satisfy the usual Heisenberg canonical commutation .

Moreover, for the de-Sitter space, it can be constructed by replacing $\alpha \longrightarrow (-\alpha')$ where α' is a positive parameter, we have $(\Delta X)(\Delta P) \geq \frac{1}{2}\left(1-\alpha'(\Delta X)^2\right)$.

While the representations of $\hat{X}$ and $\hat{P}$ can be thought of as,

$$\hat{X} = x; \quad \hat{P} = -\imath\left(1-\alpha' x^2\right)\frac{\partial}{\partial x}. \tag{4.4}$$

We note that the momentum operator $\hat{P}$ is not symmetric in all Hilbert space $L^2(\mathbb{R}, dx)$. For this, we need to change this space into subspace $L^2\left(\mathbb{R},\ d_\alpha x = \frac{dx}{1+\alpha x^2}\right)$.This makes the modified scalar product of two functions $\psi(x)$ and $\varphi(x)$ in position space basis $\{|x\rangle\}$ as

$$\langle \varphi \mid \psi \rangle = \int \varphi^*(x)\,\psi(x)\,d_\alpha x. \tag{4.5}$$

From this modification, we can construct the closure relation as follows

$$\int_{-\infty}^{+\infty} d_\alpha x\,|x\rangle\langle x| = 1, \tag{4.6}$$

and the corresponding projection relation is

$$\langle x|\,x'\rangle = \left(1+\alpha x^2\right)\delta(x-x'), \tag{4.7}$$

otherwise

$$\langle x|\,x'\rangle = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\exp\left[\frac{\imath p}{\sqrt{\alpha}}\left(\arctan\sqrt{\alpha}x - \arctan\sqrt{\alpha}x'\right)\right]. \tag{4.8}$$

Then we use the simplified form

$$\left\langle x\left|\hat{P}\right|x'\right\rangle = -\int_{-\infty}^{+\infty}\frac{pdp}{2\pi}\exp\left[\frac{\imath p}{\sqrt{\alpha}}\left(\arctan\sqrt{\alpha}x - \arctan\sqrt{\alpha}x'\right)\right]. \tag{4.9}$$

Assuming no deformation for the time component, we have

$$\langle x_0|\,x_0'\rangle = \delta(x_0 - x_0'), \quad \int_{-\infty}^{+\infty} dx_0\,|x_0\rangle\langle x_0| = 1. \tag{4.10}$$

In the following section, we concentrate on the explicit calculation of the Green function for relativistic Dirac oscillators in the context of the EUP, by using the path integral formalism.

4.3 Dirac oscillator

As it is known, the Dirac oscillator propagator in $(1+1)$ dimensions is the causal Green function $S^{(c)}(x_b, x_a)$ of the Dirac oscillator equation, which is defined as

$$\left(\gamma^{\mu}\hat{\Pi}_{b\mu}-m\right) S^{(AdS)}\left(x_b^{\mu}, x_a^{\mu}\right) = -\left(1+\alpha x_b^2\right)\delta\left(x_b - x_a\right)\delta\left(t_b - t_a\right), \tag{4.11}$$

where the components of $\hat{\Pi}_{\mu}$ are expressed as

$$\hat{\Pi}_0 = \hat{P}_0,\ \hat{\Pi}_1 = \hat{P} - \imath m\omega\gamma^0\hat{X}. \tag{4.12}$$

Here the operators $(\hat{X}, \hat{P})$ satisfy the commutation relations of the EUP, which is defined in the relation (4.1). While γ_{μ} are the Dirac matrices verify the commutation relation $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$, with the metric $\eta^{\mu\nu}$ =diag$(1, -1)$ and $\mu, \nu = 0, 1$. These Dirac matrices can be chosen in terms of Pauli matrices σ^i as follows :

$$\gamma^1 = i\sigma^1 \quad \text{and} \quad \gamma^0 = \sigma^3. \tag{4.13}$$

The corresponding solution of Eq. (4.11) is defined as the inverse of the Dirac operator $\mathcal{O}_-^d$,

$$S^{(AdS)}\left(x_b, x_a\right) = -\left\langle x_b \left| \left[\mathcal{O}_-^d\right]^{-1} \right| x_a \right\rangle = -\left\langle x_b \left| \mathcal{O}_+^d \left[\mathcal{O}_-^d \mathcal{O}_+^d\right]^{-1} \right| x_a \right\rangle. \tag{4.14}$$

$\mathcal{O}_{\pm}^d$ are defined as $\gamma^{\mu}\hat{\Pi}_{\mu} \pm m$ and $\mathcal{O}_+^d$ represents to the global Dirac projection operator. Now, following [138], the global representation for the causal Green function is obtained by inserting the completeness relation for the space-time states given by Eqs. (4.6) and (4.10) between the operators $\mathcal{O}_+^d$ and $\left[\mathcal{O}_-^d\mathcal{O}_+^d\right]^{-1}$, we get

$$S^{(AdS)}\left(x_b,x_a\right)=\left(\gamma^\mu\hat{\Pi}_\mu+m\right)_b G^{(AdS)}\left(x_b,x_a\right). \tag{4.15}$$

Here $G^{(AdS)}\left(x_b,x_a\right)$ is the global Green function defined as

$$G^{(AdS)}\left(x_b,x_a\right)=-\left\langle x_b\left|\left[\mathcal{O}_-^d\mathcal{O}_+^d\right]^{-1}\right|x_a\right\rangle, \tag{4.16}$$

and introducing the Schwinger proper-time method, the Green function $G^{(g)}\left(x_b^\mu,x_a^\mu\right)$ becomes

$$G^{(AdS)}\left(x_b^\mu,x_a^\mu\right)=\imath\int_0^{+\infty}d\lambda\left\langle x_b^\mu\left|\exp\left(\imath\lambda\mathcal{H}\right)\right|x_a^\mu\right\rangle. \tag{4.17}$$

$\mathcal{H}$ is the Hamiltonian of the system in question whose quadratic form, and will reduce to

$$\mathcal{H}=\mathcal{O}_-^d\mathcal{O}_+^d=\hat{P}_0^2-\hat{P}^2-m^2\omega^2\hat{X}^2-m^2+m\omega\gamma^0\left(1+\alpha\hat{X}^2\right), \tag{4.18}$$

which is associated the case of anti-de Sitter space and their corresponding representation (4.3). Then by taking into account the properties of the following exponential matrix, will simplify to :

$$\exp\left[\imath\lambda m\omega\gamma^0\left(1+\alpha\hat{X}^2\right)\right]=\cos\left(\lambda m\omega\left(1+\alpha\hat{X}^2\right)\right)+\imath\gamma^0\sin\left(\lambda m\omega\left(1+\alpha\hat{X}^2\right)\right), \tag{4.19}$$

or in another form

$$\exp\left[\imath\lambda m\omega\gamma^0\left(1+\alpha\hat{X}^2\right)\right]=\sum_{s=\pm1}\chi_s\chi_s^\dagger\,\exp\left[\imath\lambda sm\omega\left(1+\alpha\hat{X}^2\right)\right], \tag{4.20}$$

where $\chi_s^{\dagger}=\frac{1}{2}\left(\begin{array}{cc} 1+s, & 1-s \end{array}\right)$. Substituting (4.20) into (4.17), the global Green function can write as follow

$$G^{(AdS)}\left(x_b^{\mu},x_a^{\mu}\right)=\sum_{s=\pm 1}\chi_s\chi_s^{\dagger}\mathcal{G}^{(AdS)}\left(x_b^{\mu},x_a^{\mu}\right). \tag{4.21}$$

$\mathcal{G}^{(dS)}\left(x_b^{\mu},x_a^{\mu}\right)$ is the new global representation defined by :

$$\mathcal{G}^{(AdS)}\left(x_b^{\mu},x_a^{\mu}\right)=\imath\int_0^{+\infty}d\lambda\left\langle x_b^{\mu}\left|\exp\left(-\imath\lambda\mathcal{H}^{(s)}\right)\right|x_a^{\mu}\right\rangle, \tag{4.22}$$

and

$$\mathcal{H}^{(s)}=-\left[\hat{P}_0^2-\hat{P}^2-m^2-(m\omega)^2\hat{X}^2+sm\omega\left(1+\alpha\hat{X}^2\right)\right]. \tag{4.23}$$

At this level, to derive a path integral representation for $\mathcal{G}^{(AdS)}\left(x_b^{\mu},x_a^{\mu}\right)$, we follow the standard discretization method, we write $\exp\left(-\imath\lambda\mathcal{H}^{(s)}\right)$ by $\left[\exp\left(-\imath\lambda\mathcal{H}^{(s)}/\left(N+1\right)\right)\right]^{N+1}$, and we insert N times the identities of Eqs. (4.6) and (4.10) between each pair of operators $\exp\left(-\imath\varepsilon\lambda\mathcal{H}^{(s)}\right)$ infinitesimal with $\varepsilon=1/\left(N+1\right)$. Then, taking at the end, the limit $N\rightarrow\infty$, the expression of $\mathcal{G}^{(AdS)}\left(x_b^{\mu},x_a^{\mu}\right)$ becomes as,

$$G^{(AdS)}\left(x_b^{\mu},x_a^{\mu}\right)=\text{-I}\lim_{N\rightarrow\infty}\int_0^{\infty}\text{D}\lambda\prod_{j=1}^{N}\left[\int d_{\alpha}x_j dx_{0j}\right]\prod_{j=1}^{N+1}\langle x_j,x_{0j}|\left[1-\imath\varepsilon\lambda\mathcal{H}^{(s)}+O\left(\varepsilon^{n\geqslant 2}\right)\right]|x_{j-1}x_{0j-1}\rangle, \tag{4.24}$$

where $x_0=x_a, x_{00}=x_{0a}, x_{N+1}=x_b$ and $x_{0N+1}=x_{0b}$. Using the relations (4.3) and (4.9) into (4.24), and introducing the integral representation follow :

$$\langle x_j,x_{0j}\mid x_{j-1},x_{0j-1}\rangle=\int\frac{dp_{0j}}{2\pi}\exp\left(\imath p_{0j}\Delta x_{0j}\right)\int\frac{dp_j}{2\pi}\exp\left(\imath p_j\frac{\Delta\arctan\left(\sqrt{\alpha}x_j\right)}{\sqrt{\alpha}}\right), \tag{4.25}$$

we will get the following expression

$$\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right) = -\imath \lim_{N\to\infty} \int_0^\infty d\lambda \prod_{j=1}^{N} \left[\int d_\alpha x_j dx_{0j}\right] \prod_{j=1}^{N+1} \left[\int \frac{dp_j}{2\pi}\frac{dp_{0j}}{2\pi}\right]$$
$$\times \exp\left\{\imath \sum_{j=1}^{N+1}\left[p_j \frac{\Delta \arctan\left(\sqrt{\alpha}x_j\right)}{\sqrt{\alpha}} + p_{0j}\Delta x_{0j} + \varepsilon\lambda\left(p_{0j}^2\right.\right.\right.$$
$$\left.\left.\left.-p_j^2 - (m\omega)^2 x_j^2 - m^2 + sm\omega\left(1+\alpha x_j^2\right)\right)\right]\right\}. \qquad (4.26)$$

The Gaussian integration on the p_j, x_{0j} and p_{0j} variables is immediate, we obtain

$$\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right) = -\imath \lim_{N\to\infty} \int_0^\infty d\lambda \int \frac{dp_0}{2\pi} e^{ip_0(x_{0b}-x_{0a})} \prod_{j=1}^{N} \int d_\alpha x_j \prod_{j=1}^{N+1} \frac{1}{\sqrt{4\pi\imath\lambda\varepsilon}}$$
$$\times \exp\left\{\imath \sum_{j=1}^{N+1}\left[\frac{\left(\Delta \arctan\left(\sqrt{\alpha}x_j\right)\right)^2}{4\lambda\varepsilon\alpha} + \varepsilon\lambda\left(p_0^2 - (m\omega)^2 x_j^2 - m^2 + sm\omega\left(1+\alpha x_j^2\right)\right)\right]\right\}. \quad (4.27)$$

It is remarkable that our system converted to the case of the position-dependent effective mass. Now, in order to make this expression to the ordinary form of the Feynman path integral with constant mass, we will use the following coordinate transformation method,

$$y = f(x), \quad y_0 = x_0. \qquad (4.28)$$

This new $y-$variable changes in the interval $\left]-\frac{\pi}{2\sqrt{\alpha}}, +\frac{\pi}{2\sqrt{\alpha}}\right[$ according to variables of x in the interval $]-\infty, +\infty[$. In order to determine all quantum fluctuations, we perform the corrections associated with measure and action terms : $\left(dx_j/(1+\alpha x_j^2)\right)$ and $\left(\left(\Delta \arctan\left(\sqrt{\alpha}x_j\right)\right)^2/4\lambda\varepsilon\alpha\right)$ to get the conventional form of Feynman path integral. To determine the appropriate corrections and avoid any ambiguities, we discretize the measure and choose for any δ-point discretization interval $(x_j^{(\delta)} = \delta x_j + (1-\delta)\, x_{j-1})$ according to the technique used in [69]. So after straightforward calculations, we obtain the total quantum correction with two approaches, Kleinert method [139] and standard method [140]

$$C_{Kleinert}^T = 2i\varepsilon\lambda\frac{m^2\omega^2}{\alpha}\left(1+\tan^2(\sqrt{\alpha}y)\right)\delta\left(2\delta-1\right), \qquad (4.29)$$

and

$$C^T_{Khand} = 2i\varepsilon\lambda\frac{m^2\omega^2}{\alpha}\left[\left(1-8\delta+8\delta^2\right)\tan^2(\sqrt{\alpha}y)\right]. \tag{4.30}$$

In order to obtain the exact results we should give the two values of δ, when using Kleinert method [139] we will find $\delta = 0, 1/2$, and when using a standard method [140] we get $\delta = \frac{1}{2}\left(1 \pm 1/\sqrt{2}\right)$. These points are the same obtained in one dimension case with generalized uncertainty principle [69]. Under these considerations we will simplify C_T to zero, and the amplitude $\mathcal{G}^{(AdS)}\left(y^\mu_b, y^\mu_a\right)$ becomes as follows :

$$\mathcal{G}^{(AdS)}\left(y^\mu_b, y^\mu_a\right) = -\imath\int_0^\infty d\lambda\prod_{j=1}^{N}\int\frac{dp_0}{2\pi}e^{ip_0(y_{0b}-y_{0a})}\prod_{j=1}^{N+1}e^{\imath\lambda\left[p_0^2-m^2+sm\omega\right]}$$
$$\times\mathcal{K}^{(PT)}\left(y^\mu_b, y^\mu_a\right). \tag{4.31}$$

Here $\mathcal{K}^{(PT)}\left(y^\mu_b, y^\mu_a\right)$ is identical the propagator to the standard problem of the Poschl–Teller potential defined by

$$K^{(PT)}\left(y^\mu_b, y^\mu_a\right) = \lim_{N\to\infty}\prod_{j=1}^{N}\int \mathrm{DY}_j\prod_{j=1}^{N+1}\frac{1}{\sqrt{4\pi\imath\lambda\varepsilon}}\exp\left\{\imath\left[\frac{(\Delta y_j)^2}{4\lambda\varepsilon}-\varepsilon\lambda\left(\frac{m^2\omega^2}{\alpha}-sm\omega\right)\tan^2\left(\sqrt{\alpha}y_j\right)\right]\right\}. \tag{4.32}$$

While in the case of de-Sitter space we replace α by $(-\alpha)$, and it is given the standard problem of the modified Poschl-Teller potential.

4.3.1 Calculation of the propagator

The path integral of $y\left(t_j\right)$ in Eq. (4.32) (i.e., anti-de Sitter space) is exactly the propagator associated with the Poschl-Teller potential. Which is solved exactly in Refs. [140, 141], and equals

$$\mathcal{G}^{(AdS)}\left(y^\mu_b, y^\mu_a\right) = -\imath\lim_{N\to\infty}\int_0^\infty d\lambda\int\frac{dp_0}{2\pi}e^{ip_0(y_{0b}-y_{0a})}\; e^{\imath\lambda\left[p_0^2-m^2+sm\omega\right]}$$
$$\times\left[\sum_{n=0}^{\infty}e^{-\imath\lambda E^{(PT)}_{n,s}}\Psi^{(PT)}_{n,s}\left(y_b\right)\left(\Psi^{(PT)}_{n,s}\right)^*\left(y_a\right)\right], \tag{4.33}$$

with $E_{n,s}^{(PT)}$ is the energy spectrum associated to the Poschl-Teller potential, which is defined as

$$E_{n,s}^{(PT)} = \alpha \left(n^2 + (2n+1)\,\eta_s \right), \tag{4.34}$$

and $\psi_{n,s}^{(PT)}(y)$ are corresponding the wave functions and given by

$$\psi_{n,s}^{(PT)}(y) = \Gamma(\eta_s) \sqrt{\frac{2^{2\eta_s-1} n!\,(n+\eta_s)\sqrt{\alpha}}{\pi \Gamma(n+2\eta_s)}} \left(\cos\left(\sqrt{\alpha} y\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha} y\right)\right), \tag{4.35}$$

where

$$\eta_s = \frac{1-s}{2} + \frac{m\omega}{\alpha}. \tag{4.36}$$

η_s is parameter determined according the condition of the extended uncertainty principle, and the characteristic length of oscillator, retaining the solution associated with $\frac{m\omega}{\alpha} > 1$, the values $\eta_+ = \frac{m\omega}{\alpha}$, $\eta_- = 1 + \frac{m\omega}{\alpha}$ are accepted. While the other negatives values are rejected. Substituting (4.33) into (4.21), the propagator becomes as :

$$\begin{aligned} G^{(AdS)}\left(y_b^\mu, y_a^\mu\right) = &-\imath \int_0^\infty d\lambda \int \frac{dp_0}{2\pi} e^{ip_0(y_{0b}-y_{0a})} \sum_n \sum_{s=\pm1} \chi_s \chi_s^\dagger e^{\imath\lambda\left[p_0^2 - m^2 + sm\omega - E_{n,s}^{(PT)}\right]} \\ &\times \left[\left(\Gamma(\eta_s)\right)^2 \frac{2^{2\eta_s-1} n!\,(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)} \left(\cos\left(\sqrt{\alpha} y_b\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha} y_b\right)\right) \right. \\ &\left. \times \left(\cos\left(\sqrt{\alpha} y_a\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha} y_a\right)\right) \right]. \end{aligned} \tag{4.37}$$

An integration over λ to give this expression

$$\begin{aligned} G^{(AdS)}\left(y_b^\mu, y_a^\mu\right) = &-\imath \sum_n \sum_{s=\pm1} \left(\Gamma(\eta_s)\right)^2 \frac{2^{2\eta_s-1} n!\,(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)} \int \frac{dp_0}{2\pi} \frac{e^{ip_0(y_{0b}-y_{0a})}}{p_0^2 - \mathcal{E}_{n,s}} \chi_s \chi_s^\dagger \\ &\times \left[\left(\cos\left(\sqrt{\alpha} y_b\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha} y_b\right)\right) \left(\cos\left(\sqrt{\alpha} y_a\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha} y_a\right)\right) \right], \end{aligned} \tag{4.38}$$

with

$$\mathcal{E}_{n,s}(p_0) = m^2 - sm\omega + E_{n,s}^{(PT)}. \tag{4.39}$$

The above equation (4.38) lacks the integration over energy p_0. This can be converted to a complex integration along the special contour C, and then using the residue theorem, we have :

$$\oint \frac{dp_0}{2\pi i} \frac{e^{ip_0(y_{0b}-y_{0a})}}{p_0^2 - \mathcal{E}_{n,s}} f(p_0) = \sum_{k=1}^{n} Res\left(\frac{e^{-iE(t_b-t_a)}}{E^2-\mathcal{E}_{n,s}} f(E), E_k\right) e^{-iE_k(y_{0b}-y_{0a})}$$
$$= \sum_{\epsilon=\pm 1} \frac{f\left(E_{n,s}^{(\epsilon)}\right)}{2\epsilon\omega_{n,s}^{(AdS)}} e^{-iE_{n,s}^{(\epsilon)}(y_{0b}-y_{0a})} \Theta\left(\epsilon\left(y_{0b}-y_{0a}\right)\right), \tag{4.40}$$

where $\epsilon = \pm 1$ and $\Theta(x)$ is the Heaviside function. This gives the following poles :

$$E_{n,s}^{(\epsilon)} = \epsilon\omega_{n,s}^{(AdS)} = \pm\sqrt{m^2 - sm\omega + \alpha\left(n^2 + (2n+1)\eta_s\right)}. \tag{4.41}$$

Using the residue theorem on global Green function expression defined in Eq. (4.38). The integrations over p_0 are carried, and becomes as

$$G^{(AdS)}\left(y_b^\mu, y_a^\mu\right) = -\imath \sum_{\epsilon=\pm 1}\sum_{s=\pm 1}\sum_{n} \left(\Gamma\left(\eta_s\right)\right)^2 \frac{2^{2\eta_s-1} n! \left(n+\eta_s\right)\sqrt{\alpha}}{\pi\Gamma\left(n+2\eta_s\right)}$$
$$\times \left\{ \frac{e^{-i\epsilon\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\epsilon\omega_{n,s}^{(AdS)}} \Theta\left(\epsilon\left(y_{0b}-y_{0a}\right)\right) \chi_s\chi_s^\dagger \left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\right.$$
$$\left. \times \left[\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right]. \right. \tag{4.42}$$

Furthermore, we can get the global Green function $G^{(dS)}\left(y_b^\mu, y_a^\mu\right)$ in de-Sitter Snyder case, just by changing α by $(-\alpha)$ with retaining the term of η_s.

4.3.2 Spectral energies and Spinorial wave functions

To obtain the exact solutions for the wave functions and spectral energies for the system governed by the Dirac equation, it must bring the corresponding spectral decomposition of Dirac oscillator in (1+1) dimension in the context of the EUP by the act operator $(\gamma^{\nu}\hat{\Pi}_{\nu}+m)_b$ on Eq. (4.42). This will be simplified as

$$
\begin{aligned}
S^{(AdS)}(x_b,x_a) &= -\imath\left[\imath\sigma^3\frac{\partial}{\partial y_{0b}}+\sigma^1\left(\frac{\partial}{\partial y_b}+\sigma^3\frac{m\omega}{\sqrt{\alpha}}\tan\left(\sqrt{\alpha}y_b\right)\right)+m\right] \\
&\times\sum_{s=\pm1}\sum_{n}\Bigg\{(\Gamma(\eta_s))^2\,\frac{2^{2\eta_s-1}n!\,(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}\frac{e^{-is\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\omega_{n,s}^{(AdS)}}\Theta\left(s\left(y_{0b}-y_{0a}\right)\right)\chi_s\chi_s^{\dagger} \\
&\times\left[\left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right]\Bigg\} \\
&+\Bigg\{-(\Gamma(\eta_s))^2\,\frac{2^{2\eta_s-1}n!\,(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}\frac{e^{+is\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\omega_{n,s}^{(AdS)}}\Theta\left(-s\left(\left(y_{0b}-y_{0a}\right)\right)\right)\chi_s\chi_s^{\dagger} \\
&\times\left[\left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right]\Bigg\}. \qquad (4.43)
\end{aligned}
$$

With some known relationships in algebra matrices for Dirac, we have

$$\sigma^3\chi_s = s\chi_s,\quad \sigma^1\chi_s=\chi_{-s}\ \text{ and }\ \sigma^2\chi_s = is\chi_{-s}, \qquad (4.44)$$

and with helping of Gegenbauer's polynomials properties [142],

$$
\begin{cases}
\frac{d}{du}C_n^{\eta}(u) = 2\eta C_{n-1}^{\eta+1}(u)\,, \\
nC_n^{\eta}(u) = (2\eta+n-1)uC_{n-1}^{\eta}(u) - 2\eta(1-u^2)C_{n-2}^{\eta+1}(u)\,, \\
(2\eta+n)\,C_n^{\eta}(u) = 2\eta\left[C_n^{\eta+1}(u) - uC_{n-1}^{\eta+1}(u)\right],
\end{cases} \qquad (4.45)
$$

we can write the Green function through a straightforward calculation, as follows :

$$
\begin{aligned}
\mathrm{S}^{(AdS)}\left(x_b, x_a\right) = &-\mathrm{I} \sum_{s=\pm 1} \sum_{n} \left(\Gamma\left(\eta_s\right)\right)^2 \frac{2^{2\eta_s - 1} n! \left(n+\eta_s\right) \sqrt{\alpha}}{\pi \Gamma\left(n+2\eta_s\right)} \left(\cos\left(\sqrt{\alpha} y_b\right)\right)^{\eta_s} \left(\cos\left(\sqrt{\alpha} y_a\right)\right)^{\eta_s} \\
&\times \left\{ \frac{e^{-is\omega_{n,s}^{(AdS)}\left(y_{0b}-y_{0a}\right)}}{2\omega_{n,s}^{(AdS)}} \Theta\left(s\left(y_{0b}-y_{0a}\right)\right) \left[\left(\omega_{n,s}^{(dS)} + m\right) C_n^{\eta_s}\left(\xi_b\right) C_n^{\eta_s}\left(\xi_a\right) \chi_s \chi_s^{\dagger} \right.\right. \\
&\left.\left. + \sqrt{\alpha} \left[-\left(\frac{1-s}{2}\right) \left(1+\frac{m\omega}{\alpha}\right) \tan\left(\sqrt{\alpha} y_b\right) C_n^{\eta_s}\left(\xi_b\right) + 2\eta_s \cos\left(\sqrt{\alpha} y_b\right) C_{n-1}^{\eta_s+1}\left(\xi_b\right) \right] C_n^{\eta_s}\left(\xi_a\right) \chi_{-s} \chi_s^{\dagger} \right] \right\} \\
&- \left\{ \frac{e^{+is\omega_{n,s}^{(AdS)}\left(y_{0b}-y_{0a}\right)}}{2\omega_{n,s}^{(AdS)}} \Theta\left(-s\left(y_{0b}-y_{0a}\right)\right) \left[\left(-\omega_{n,s}^{(dS)} + m\right) C_n^{\eta_s}\left(\xi_b\right) C_n^{\eta_s}\left(\xi_a\right) \chi_s \chi_s^{\dagger} \right.\right. \\
&\left.\left. + \sqrt{\alpha} \left[-\left(\frac{1-s}{2}\right) \left(1+\frac{m\omega}{\alpha}\right) \tan\left(\sqrt{\alpha} y_b\right) C_n^{\eta_s}\left(\xi_b\right) + 2\eta_s \cos\left(\sqrt{\alpha} y_b\right) C_{n-1}^{\eta_s+1}\left(\xi_b\right) \right] C_n^{\eta_s}\left(\xi_a\right) \chi_{-s} \chi_s^{\dagger} \right] \right\}.
\end{aligned}
\tag{4.46}
$$

Now, to obtain the spectral energies and corresponding eigenfunctions, we must unify the expression of energy $E_{n,s}^{(\epsilon)}$. Which leads us to make the following changes on the second term in the Green function, which are multiplied by $\Theta\left(-s\left(y_{0b}-y_{0a}\right)\right)$

$$
\begin{aligned}
s &\rightarrow s' = -s, \\
n &\rightarrow n' = n - s, \\
\eta_s &\rightarrow \eta_{s'} = \eta_s + s.
\end{aligned}
\tag{4.47}
$$

After these changes, we can write

$$S^{(AdS)}\left(x_b,x_a\right)=-\imath\sum_{s=\pm1}\sum_{n}\left(\Gamma\left(\eta_s\right)\right)^2\frac{2^{2\eta_s-1}n!\left(n+\eta_s\right)\sqrt{\alpha}}{\pi\Gamma\left(n+2\eta_s\right)}$$
$$\times\left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s}\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s}\frac{e^{-is\omega_{n,s}^{(dS)}\left(y_{0b}-y_{0a}\right)}}{2\omega_{n,s}^{(dS)}}\Theta\left(s\left(y_{0b}-y_{0a}\right)\right)$$
$$\times\left\{\left[\left(\omega_{n,s}^{(dS)}+m\right)C_n^{\eta_s}\left(\xi_b\right)C_n^{\eta_s}\left(\xi_a\right)\chi_s\chi_s^{\dagger}+\right.\right.$$
$$+\sqrt{\alpha}\left[-\left(\frac{1-s}{2}\right)\left(1+\frac{m\omega}{\alpha}\right)\tan\left(\sqrt{\alpha}y_b\right)C_n^{\eta_s}\left(\xi_b\right)\right.$$
$$\left.\left.+2\left(\frac{1-s}{2}+\frac{m\omega}{\alpha}\right)\cos\left(\sqrt{\alpha}y_b\right)C_{n-1}^{\eta_s+1}\left(\xi_b\right)\right]C_n^{\eta_s}\left(\xi_a\right)\chi_{-s}\chi_s^{\dagger}\right\}$$
$$-\left\{\left[\left(-\omega_{n,s}^{(dS)}+m\right)C_{n-s}^{\eta_s+s}\left(\xi_b\right)C_{n-s}^{\eta_s+s}\left(\xi_a\right)\chi_{-s}\chi_{-s}^{\dagger}\right.\right.$$
$$+\sqrt{\alpha}\left[-\left(\frac{1+s}{2}\right)\left(1+\frac{m\omega}{\alpha}\right)\tan\left(\sqrt{\alpha}y_b\right)C_{n-s}^{\eta_s+s}\left(\xi_b\right)\right.$$
$$\left.\left.+2\left(\frac{1+s}{2}+\frac{m\omega}{\alpha}\right)\cos\left(\sqrt{\alpha}y_b\right)C_{n-s-1}^{\eta_s+s+1}\left(\xi_b\right)\right]C_{n-s}^{\eta_s+s}\left(\xi_a\right)\chi_s\chi_{-s}^{\dagger}\right\}. \quad (4.48)$$

From above expression, we can rewrite the causal Green's function as follows :

$$S^{(AdS)}\left(x_b,x_a\right)=-\imath\sum_{s=\pm1}\sum_{n}\exp\left(-\imath s\omega_{n,s}^{(AdS)}\left(y_{0b}-y_{0a}\right)\right)\Theta\left(s\left(y_{0b}-y_{0a}\right)\right)$$
$$\times\left[\Gamma\left(\eta_s\right)\sqrt{\tfrac{2^{2\eta_s-1}n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}}\sqrt{\tfrac{m+\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}}C_n^{\eta_s}\left(u_b\right)\vartheta_b^{\eta_s}\chi_s\right.$$
$$\left.+\imath\sqrt{\alpha}\Gamma\left(\eta_s+s\right)\sqrt{\tfrac{2^{2(\eta_s+s)-1}(n-s)!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s+s)}}\sqrt{\tfrac{m-\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}}C_{n-s}^{\eta_s+s}\left(u_b\right)\vartheta_b^{\eta_s+s}\chi_{-s}\right]$$
$$\times\left[\Gamma\left(\eta_s\right)\sqrt{\tfrac{2^{2\eta_s-1}n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}}\sqrt{\tfrac{m+\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}}C_n^{\eta_s}\left(u_a\right)\vartheta_a^{\eta_s}\chi_s^{\dagger}\right.$$
$$\left.+\imath\sqrt{\alpha}\Gamma\left(\eta_s+s\right)\sqrt{\tfrac{2^{2(\eta_s+s)-1}(n-s)!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s+s)}}\sqrt{\tfrac{m-\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}}C_{n-s}^{\eta_s+s}\left(u_a\right)\vartheta_a^{\eta_s+s}\chi_{-s}^{\dagger}\right]. \quad (4.49)$$

In Eq. (4.49) we have two types of propagation, one with positive energy ($+E_{n,\alpha}^{Anti}$) propagating to the future and the other with negative energy ($-E_{n,\alpha}^{Anti}$) propagating to the past. Consequently, we obtain this result in the former variable,

$$S^{(\alpha)}\left(x_b, x_a, t_b - t_a\right) = -\sum_{s=\pm 1}\sum_{n}\left[\begin{array}{c}\Theta\left(t_b - t_a\right)\Psi_n^{(\alpha)+}\left(x_b\right)\bar{\Psi}_n^{(\alpha)+}\left(x_a\right)e^{-\imath E_{n,\alpha,s}^{Anti}\left(t_b - t_a\right)} + \\ \Theta\left(-\left(t_b - t_a\right)\right)\Psi_n^{(\alpha)-}\left(x_b\right)\bar{\Psi}_n^{(\alpha)-}\left(x_a\right)e^{\imath E_{n,\alpha,s}^{Anti}\left(t_b - t_a\right)}\end{array}\right]. \tag{4.50}$$

This formula is the spectral decomposition of the Green function, within which we extract the wave functions

$$\begin{aligned}\Psi_n^{(AdS)s}(x) &= \Gamma\left(\eta_s\right)\sqrt{\tfrac{2^{2\eta_s - 1} n!\left(n+\eta_s\right)\sqrt{\alpha}}{\pi\Gamma\left(n+2\eta_s\right)}}\sqrt{\tfrac{m+\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_n^{\eta_s}\left(u_b\right)\vartheta_b^{\eta_s}\chi_s \\ &+ \imath\sqrt{\alpha}\Gamma\left(\eta_s + s\right)\sqrt{\tfrac{2^{2\left(\eta_s + s\right) - 1}(n-s)!\left(n+\eta_s\right)\sqrt{\alpha}}{\pi\Gamma\left(n+2\eta_s + s\right)}}\sqrt{\tfrac{m-\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_{n-s}^{\eta_s + s}\left(u_b\right)\vartheta_b^{\eta_s + s}\chi_{-s},\end{aligned} \tag{4.51}$$

and we can return to the old variables by means of the following relations

$$u = \sin\left(\arctan\left(\sqrt{\alpha}x\right)\right),\ \vartheta = \cos\left(\arctan\left(\sqrt{\alpha}x\right)\right). \tag{4.52}$$

Where the corresponding spectral energies are

$$E_{n,\alpha,s}^{(AdS)\pm} = \pm\sqrt{m^2 + 2m\omega\left(n + \frac{1-s}{2}\right) + \alpha\left(n^2 + \frac{1-s}{2}\left(2n+1\right)\right)}. \tag{4.53}$$

The dependence on n^2 corresponding to α effect of the modification of the Heisenberg algebra, due to the EUP, which is a characteristic of the confinement phenomena. With various the values of α and with spin up ($s = 1$), we can plot the appropriate curves of positive and negative energies in Fig. 1. We clearly notice that the energy $E_{n,\alpha,s}^{(AdS)\pm}$ is presented as a function of n for several values of α, the spectrum is expanded, $E_{n,\alpha,s}^{(AdS)+}$

is an increasing ($E_{n,\alpha,s}^{(AdS)-}$ is decreasing) monotonous function for arbitrary α.

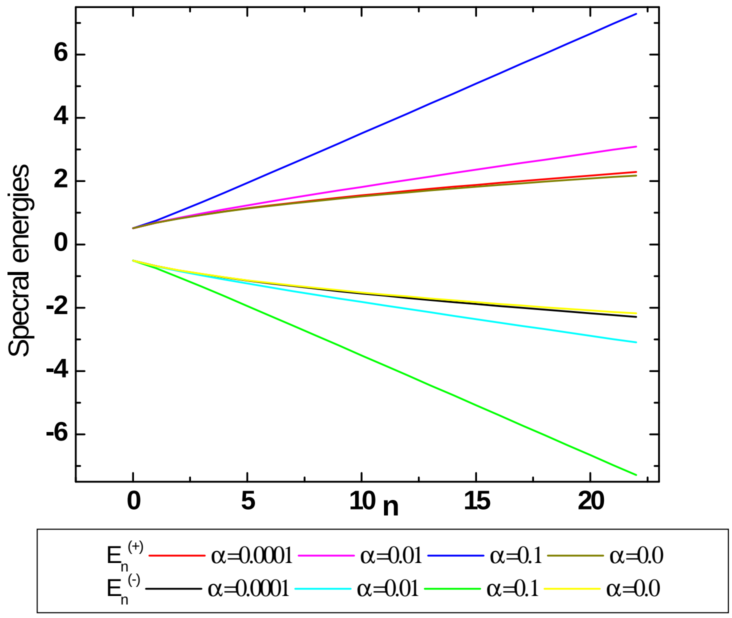

Fig. 9. $E_{n,s=1}^{(AdS)}$ is the energy spectrum versus n for several values of α

Next, we want to check the current density (ρ, J_x) for $(1+1)$ −dimensional Dirac oscillator in the context of the EUP. Activating the positive use of this method (path integral formalism) for normalized the wave functions in the context of the extended uncertainty principle. As we know the current density are defined as

$$\rho = \int d_\lambda x (\Psi_n^{(AdS)s}(x))^\dagger \Psi_n^{((AdS))s}(x), \tag{4.54}$$

$$J_x = \int d_\lambda x \bar{\Psi}_n^{((AdS))s}(x)\, \gamma^1 \Psi_n^{((AdS))s}(x). \tag{4.55}$$

After straightforward calculation, we can confirm the current density of Dirac oscillator in (1+1) dimension in the context of the EUP are given as

$$\rho = 1, \tag{4.56}$$

$$J_x = \int d_\lambda x(\Psi_n^{(AdS)s}(x))^\dagger \sigma^2 \Psi_n^{(AdS)s}(x) = 0. \tag{4.57}$$

Which approves the same results in usual case of the Dirac oscillator in (1+1) dimension ($\alpha = 0$).

4.3.3 de Sitter Snyder spaces

In the case of de-Sitter Snyder space, we will follow the same calculation procedures as presented in the previous section. Which can be constructed by replacing α by $(-\alpha')$ in Eq. (4.53). The spectral energies $E_{n,\alpha',s}^{(dS)\pm}$ are given as :

$$E_{n,\alpha',s}^{(dS)\pm} = \pm\sqrt{m^2 + 2m\omega\left(n + \frac{1-s}{2}\right) - \alpha'\left(n^2 + \frac{1-s}{2}(2n+1)\right)}. \tag{4.58}$$

We note that when the quantum number n is large, the spectral energies would have no physical meaning. This indicates that one needs to impose an upper bound on the values of n. From these last expressions of the spectral energies $E_{n,\alpha',s}^{(dS)\pm}$, we can determine this limit by using

$$\frac{E_{n,\alpha',s}^{(dS)}}{dn} |_N = 0, \tag{4.59}$$

where N implies to $\left(\frac{m\omega}{\alpha'}+\frac{1-s}{2\alpha'}\right)$, and $E^{(dS)+}_{n,\alpha',s}$ is decreasing (while $E^{(dS)-}_{n,\alpha',s}$ is an increasing) monotonous function for arbitrary α'. These cases are illustrated by the following curve

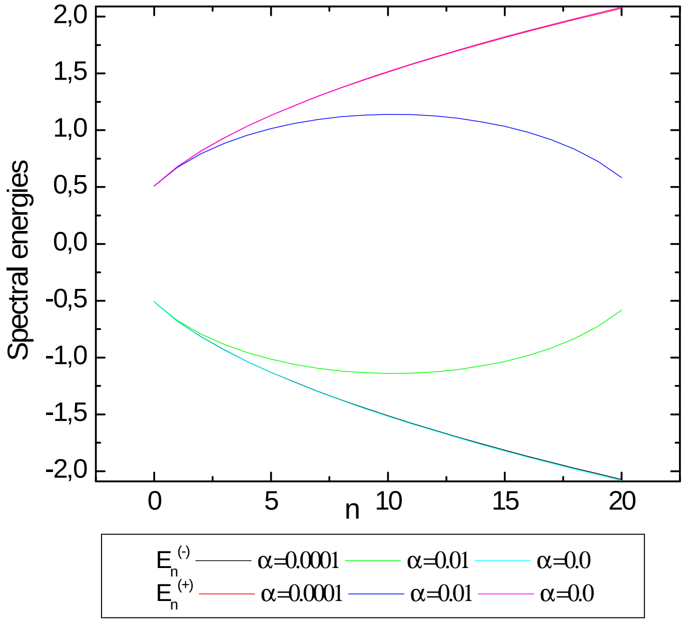

Fig. 10. $E^{(dS)}_{n,s=1}$ is the energy spectrum versus n for several values of α'

While the corresponding wave functions are given from Eq. (4.51) by substituting ($\alpha \to -\alpha'$), and which leads to

$$u=\sinh\left(\tanh^{-1}\left(\sqrt{\alpha'}x\right)\right),\ \vartheta=\cosh\left(\tanh^{-1}\left(\sqrt{\alpha'}x\right)\right). \tag{4.60}$$

In the following subsections, we will present the special and important cases to validate these our calculations.

Without deformation case

In order to obtain the ordinary case, we put the limit $\alpha \to 0$, $\eta_s = \eta \to \infty$, by using [142]

$$\lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} C_n^{\frac{\lambda}{2}} \left(x\sqrt{\frac{2}{\lambda}} \right) = \frac{2^{-\frac{n}{2}}}{n!} H_n(x), \qquad \lim_{\lambda\to\infty} \frac{\Gamma(\lambda+a)}{\Gamma(\lambda)} e^{-a\ln\lambda} = 1, \tag{4.61}$$

the doubling formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x) \Gamma\left(x + \frac{1}{2} \right), \tag{4.62}$$

and

$$\lim_{\alpha\to 0, \eta\to\infty} \left(1+\alpha x^2\right)^{\eta} = \exp\left(-\frac{m\omega}{2} x^2\right). \tag{4.63}$$

From the above limits, we can obtain the wave functions and energy spectrum, and they are given respectively

$$\lim_{\alpha\longrightarrow 0} \Psi_n^{(\alpha)s}(x) = \Psi_n^{(\alpha=0)s}(x) = \begin{pmatrix} f_n^{(\alpha=0)s}(x) \\ g_n^{(\alpha=0)s}(x) \end{pmatrix},$$

with the following components,

$$f_n^{(\alpha=0)s}(x) = \sqrt{\frac{\sqrt{m\omega/\pi}\left(E_{n,s}^{(\alpha=0)} + m\right)}{2^{n+1} n! E_{n,s}^{(\alpha=0)}}} \exp\left(-\frac{m\omega}{2} x^2\right) H_n\left(\sqrt{m\omega} x\right). \tag{4.64}$$

$$g_n^{(\alpha=0)s}(x) = -\sqrt{\frac{\sqrt{m\omega/\pi}\left(E_{n,s}^{(\alpha=0)} - m\right)}{2^{n} (n-s)! E_{n,s}^{(\alpha=0)}}} \exp\left(-\frac{m\omega}{2} x^2\right) H_{n-s}\left(\sqrt{m\omega} x\right), \tag{4.65}$$

and

$$E_{n,\alpha'=0,s}^{(dS)\pm} = \pm\sqrt{m^2 + 2m\omega\left(n + \frac{1-s}{2}\right)}. \tag{4.66}$$

We deduct exactly the same result without deformed uncertainty relation which coincide with those obtained from the usual Dirac oscillator in $(1+1)$ dimensions [128].

Non -relativistic limit

To obtain the energy level in non relativistic limit case for the one-dimensional Dirac oscillator in anti-de Sitter spaces system $E_{NR,s}^{(AdS)}$, we put $E_{n,\alpha,s}^{(AdS)} = m + E_{NR,s}^{(AdS)}$ with $m \gg E_{NR,s}^{(AdS)}$ and using the Taylor development of (4.53) in the second order approximation,we find :

$$E_{NR,s}^{(AdS)+} \approx \omega\left(n+\frac{1-s}{2}\right)+\frac{\alpha}{2m}\left(n^2+\frac{1-s}{2}(2n+1)\right) + \frac{1}{2}\left[\omega\left(n+\frac{1-s}{2}\right)+\frac{\alpha}{2m}\left(n^2+\frac{1-s}{2}(2n+1)\right)\right]^2, \tag{4.67}$$

with m represents the rest energy of the particle, the second and third terms represent, respectively, the energy of the non-relativistic oscillator of frequency ω and the relativistic correction both in the context of the extended uncertainty principle.

This implies that the corresponding eigenvalues associated with this energy level in the non-relativistic limit are given by

$$\Psi_{NR,s}^{(AdS)+}(x) = \Gamma(\eta_s)\sqrt{\frac{2^{2\eta_s-1}n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}}C_n^{\eta_s}(u_b)\,\vartheta_b^{\eta_s}\chi_s, \tag{4.68}$$

where we have used the following limits :

$$\lim_{m\gg}\sqrt{\frac{m+\omega_{n,s}^{(AdS)+}}{2\omega_{n,s}^{(AdS)+}}} \approx 1, \lim_{m\gg}\sqrt{\frac{m-\omega_{n,s}^{(AdS)+}}{2\omega_{n,s}^{(AdS)+}}} \approx 0. \tag{4.69}$$

In this contribution, we have constructed the path integral representation for the Green function for the Dirac oscillator in (1+1) dimension in the EUP. Which indicates to presence a nonzero minimal uncertainty in momentum. We obtained the exact spectral energies and corresponding eigenfunctions expressed in terms of Gegenbauer polynomials. The energy levels show a dependence on n^2 corresponding to α effect of the modification of the Heisenberg algebra, due to the EUP, which is a characteristic of the confinement phenomena as in the case of non-commutative geometry. As a result, for a fixed value of

n, the energy $E_{n,\alpha,+}^{(AdS)}$ increases monotonically with the increase of the EUP parameter α.

We have also deduced special cases :

1) de-Sitter spaces case by replacing $\alpha \longrightarrow -\alpha'$, we note that the corresponding energy spectrum $E_{n,\alpha',s}^{(dS)}$ would have an unphysical behavior when the quantum number n is large. This indicates that one needs to impose an upper bound on the values of n and we can also see that the energy spectrum on the de Sitter space is smaller than the energy in ordinary quantum mechanics.

2) absence of deformation case by taking the limit ($\alpha \longrightarrow 0$), we obtain the usual Heisenberg algebra. The same result without deformed uncertainty relation which has been done by Rekioua et al [128]. A generalization of this work in the presence of an electromagnetic field that requires a thorough discussion is currently under consideration, and will be the subject of another study. At the end of this paper, it is worth mentioning that the results obtained make it possible to detect the effects due to the large scale curvature of space-time on some physical systems : for example the confinement of quarks in quantum chromodynamics (QCD) and the description of certain properties of electrons in graphene. We recall that the dynamics of these two physical examples cited is modeled by the relativistic Dirac oscillator , as it is known in the literature [134, 137].

Chapitre 5

General conclusion

This thesis essentially consists of two main parts :

- In the first part, we have established an exact and explicit solution of some relativistic problems in the context of deformed algebras via the direct method by solving the equations.

By using the new type of the extended uncertainty principle associated to the displacement operator method, several applications were presented such as :

The Klein-Gordon particle confined in a one dimensional box, the Klein Gordon equation in the presence of the linear vector and scalar potentials , the Klein Gordon in mixed Coulomb-type vector and scalar potentials and the Klein-Gordon and Dirac oscillators subject to a uniform electric field. In these cases, the exact analytical solutions are determined and the wave functions and the exact corresponding energy spectrum are extracted. It is remarkable that this deformation influences the results obtained, the expressions of energy spectrum vary with all the power of n , which explain the confinement phenomenon, it is also mentioned that the bound states are limited and the expressions of energy are not defined for large values of n. Consequently, we need to impose an upper limit on the allowed values of n. For the two last cases the Klein-Gordon and Dirac oscillators subject to a uniform electric field, we noticed that the expression of the energy spectrum contains corrections of all orders of $(\varepsilon\gamma)^2$, which could be interpreted like the Stark effect in the extended uncertainty principle deformation framework and related to the exact contribution to the confinement phenomenon .

In addition, we have treated the three-dimensional Klein-Gordon oscillator and the Klein-Gordon equation with a Coulomb plus scalar potential in the context of quantum deformations for the (anti)- de Sitter algebras . The energy eigenvalues and their corresponding eigenfunctions are exactly and analytically obtained . In both problem, we show that the spectrum energy on anti-de Sitter is bigger than the energy in ordinary case contrariwise in ds space , the energy spectrum is smaller than the energy in ordinary case.

-The second part is devoted to the elaboration of the formalism of supersymmetric path integrals in the context of deformed algebras. The Dirac oscillator in the extended uncertainty principle is exposed as being a good application. To explicitly evaluate the propagator associated with the problem, the global Dirac projection operator and the Schwinger proper-time method are introduced. To determine the appropriate corrections and avoid any ambiguities, we discretize the measure for any δ-point discretization interval. By using appropriate transformations and evaluating some Gaussian integrations, our system converted to the case of the position-dependent effective mass. To obtain the ordinary expression for the constant mass case, using a suitable coordinate transformation and by straightforward calculation the propagator will be converted to the Poschl-Teller case. Finally, the energy spectrum and the corresponding wave functions are exactly determined and are agree exactly with those in the literature. Also the limiting cases are considered .

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Scalar Particle in New Type of the Extended Uncertainty Principle

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Abstract In the context of new type of the extended uncertainty principle using the displacement operator method, we present an exact solution of some problems such as: the Klein–Gordon particle confined in a one dimensional box, the scalar particle with linear vector and scalar potentials and the case of inversely linear vector and scalar potentials of Coulomb-type. The expressions of bound state energies and the associated wave functions are exactly determined for these three cases.

1 Introduction

During the last years, there has been growing interest in studying the dynamics of quantum particles in the framework of deformed algebras. We quote some examples, the description of the low energy excitations of graphene and the Fermi velocity , is based on a deformation of the Heisenberg algebra which makes the commutator of momenta proportional to the pseudo-spin [1]. The dynamics of systems with variable masses in semiconductor heterostructures are formulated by deformed quadratic algebra [2], the thermostatistics of q-deformed bosons and fermions [3], the q-deformed quark fields [4], the motion of a ^{3}He impurity atom in the Bose liquid [5], in the context of quantum gravity, namely if we takes into account the effects of the gravitational field in order to incorporate gravity into quantum mechanics, the usual Heisenberg uncertainty principle should be replaced by the so-called generalized uncertainty principle (GUP) [6–12] and it is characterized by the existence of a minimal length scale in the order of the Planck length. The several research fields in which the concept of minimal length plays an essential role are, the string theory [13], non-commutative geometries [14], black hole physics [15] and the quantum gravity [16]. Recently, in this sense, this generalized uncertainty principle (GUP) has undergone notable development and a significant number papers have been published in diverse physics area [17–25]

In addition, if we consider the quantum effects due to the topology of the physical space, the modified uncertainty principle associated called the extended uncertainty principle (EUP) [26–31]. For example in these research works, Mignemi showed that in a (Anti) de Sitter background the Heisenberg uncertainty principle modified by introducing corrections proportional to the cosmological constant $\wedge = -3\lambda^2$, where $\lambda^2 \prec 0$ for de Sitter space-time, and $\lambda^2 \succ 0$ for anti-de Sitter space-time. The introduction of this idea of (EUP) has drawn great attention and many papers have been appeared in the literature to address the effects of the extended commutation relations [32–42]

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Furthermore, in the past few years, another new type of EUP with a minimum momentum dispersion [43], has been introduced by the action of the translation operator in a space with a diagonal metric for the purpose of describing the motion of a quantum particle in the curved space.

$$T_{\gamma}(\delta x) \mid x >=\mid x+\delta x+\gamma x \delta x > \tag{1}$$

where δx is an infinitesimal displacement and the parameter γ is the inverse of a characteristic length that determines the mixing between the displacement and the original position state [44–47]. This translation is non-additive, can be written as to first order in δx

$$T_{\gamma}(\delta x)=1-\frac{i\delta x}{\hbar}P_{\gamma}. \tag{2}$$

where P_{γ} is a generalized momentum operator. This property changes the commutation relation for position and momentum as

$$\left[\hat{x}, P_{\gamma}\right]=i\hbar(1+\gamma x), \tag{3}$$

and leads a generalized uncertainty relation

$$\Delta x \Delta P_{\gamma} \geqslant \frac{\hbar}{2}(1+\gamma\langle x\rangle). \tag{4}$$

The generalized momentum operator and the operators of position satisfying Eq. (3) can be represented in Hermitian form by [44–47]

$$P_{\gamma}=-i\hbar D_{\gamma} \text{ and } \hat{x}=x, \tag{5}$$

with

$$D_{\gamma}=\left[(1+\gamma x)\frac{d}{dx}+\frac{\gamma}{2}\right] \tag{6}$$

On the other hand, the nonadditive operator corresponds to the infinitesimal generator of the q-exponential function [48,49]

$$\exp_{q}(x) \equiv [1+(1-q)x]^{\frac{1}{1-q}}, \tag{7}$$

where x is a dimensionless variable, and $\gamma \equiv (1-q)$. This last Eq. (7) represents a fundamental mathematical definition for the generalized thermostatistics of Tsallis and its applications [50–55]. For this purpose, some problems were solved within this framework of the translation operator for a quantum system, for example, the study of particle under a null potential confined in a square well in [44–47], the solution of the quantum harmonic oscillator is given by [56,57] where the problem converted to the Morse potential case. A new Hermitian kinetic energy operator for the description of position-dependent effective mass systems void of the ordering ambiguity hassle and similar with three different ordering of the Von Roos operator is proposed by [58]. In [59], Arda used this displacement operator to study the particle moving in an inverse square plus Coulomb-like potential and it is similar the Rosen-Morse potential in usual position space.

The main purpose of this paper is to solve analytically and exactly the Klein–Gordon equation in the context of this new type of EUP for some important applications:

- Klein Gordon particle in a box model
- Klein Gordon equation with linear vector and scalar potentials
- Klein Gordon equation with inversely linear vector and scalar potentials of Coulomb-type.

To the best of our knowledge, no relativistic problem has been studied within this framework of the translation operator. Consequently, our attempt is to approach this new type of EUP for a relativistic problem and to see the influence of this deformation on the properties of the systems.

The rest of the paper is organized as follows. In Sect. 2, we give the exact solution of Klein–Gordon particle confined in a one dimensional box. The scalar particle with linear vector and scalar potentials is treated in Sect. 3. The case of Coulomb-type vector and scalar potentials has been examined in Sect. 4.

2 The Klein–Gordon Particle in One-Dimensional Box

We consider a Klein–Gordon (K–G) particle without spin of mass m and charge q confined to the following one dimensional box:

$$qV(x) = \begin{cases} 0, & 0 \leq x \leq L \\ \infty, & elsewhere \end{cases}. \tag{8}$$

So, in the context of this new type of EUP using the displacement operator method, the stationary Klein–Gordon equation in the presence of a potential $V(x)$ in one dimensional space is defined by : we put ($\hbar = c = 1$)

$$\left[(E - qV(x))^2 - P_\gamma^2 - m^2\right]\phi(x) = 0. \tag{9}$$

where P_γ is given by (5) . Moreover, the continuity equation can be deduced from the modified Klein–Gordon equation (9) and its conjugate by this relation

$$\frac{\partial \rho}{\partial t} + D_\gamma J_\gamma = 0. \tag{10}$$

with

$$\rho = \frac{i}{2m}\left(\Psi^* \partial_t \Psi - \Psi \partial_t \Psi^*\right), \tag{11}$$

and J_γ defines the modified current density

$$J_\gamma = -\frac{i}{2m}\left(\Psi^*(1+\gamma x)\frac{d\Psi}{dx} - \Psi(1+\gamma x)\frac{d\Psi^*}{dx}\right), \tag{12}$$

Now, in order to solve the Eq. (9) in one dimensional box, for $0 \leq x \leq L$, using the representation (5) and the following transformation:

$$u = (1 + \gamma x), \tag{13}$$

we obtain:

$$\left(u^2 \frac{d^2}{du^2} + 2u\frac{d}{du} + \frac{1}{4} + \frac{E^2 - m^2}{\gamma^2}\right)\phi(u) = 0, \tag{14}$$

To transform this last differential equation homogeneous to another one with constant coefficients, using the following change $z = \ln u$, we get as a result :

$$\left(\frac{d^2}{dz^2} + \frac{d}{dz} + \frac{1}{4} + \frac{E^2 - m^2}{\gamma^2}\right)\phi(z) = 0, \tag{15}$$

whose the solution in term on the old variable is given by

$$\phi(x) = \frac{\mathcal{N}}{\sqrt{(1+\gamma x)}}\sin\left(\sqrt{E^2 - m^2}\frac{\ln(1+\gamma x)}{\gamma} + \xi\right). \tag{16}$$

where $\mathcal{N}$ is a normalization constant. Using the boundary conditions $\varphi(0) = \varphi(L) = 0$, the solution of (14) will take the following form

$$\phi(x) = \frac{\mathcal{N}}{\sqrt{(1+\gamma x)}}\sin\left(\sqrt{E_n^2 - m^2}\frac{\ln(1+\gamma x)}{\gamma}\right). \tag{17}$$

with

$$\sqrt{E_n^2 - m^2}\frac{\ln(1+\gamma L)}{\gamma} = n\pi, \tag{18}$$

This gives rise to the quantized energy

$$E_n^{\pm} = \pm\sqrt{m^2 + \frac{n^2\pi^2\gamma^2}{\ln^2(1+\gamma L)}}. \tag{19}$$

Now if we consider $\gamma = 0$ absence of deformation, taking $\gamma \to 0$ in (19) we find,

$$E^{\pm} = \pm\sqrt{m^2 + \frac{n^2\pi^2}{L^2}}. \tag{20}$$

which is the result of the ordinary case [60].

The normalization constant $\mathcal{N}$ can be obtained from the normalization condition of the Ψ_n, follows from the modified definition of the scalar product for Klein Gordon equation :

$$\int_{-\infty}^{+\infty} \frac{i}{2m}\left(\Psi_n^*(x)\frac{\partial \Psi_n(x)}{\partial t} - \Psi_n(x)\frac{\partial \Psi_n^*(x)}{\partial t}\right) = 1, \tag{21}$$

and by a direct calculation, we get

$$\mathcal{N} = \sqrt{\frac{2\gamma m}{E_n \ln(1+\gamma L)}}. \tag{22}$$

3 The Klein Gordon Equation with Mixed Scalar and Vector Linear Potentials

The dynamic of Klein–Gordon particle in (1 + 1) dimension in the presence of a scalar potential $S(x)$ and a vector potential $V(x)$ in the framework of of new type of EUP is governed by this stationary equation :

$$\left[P_\gamma^2 + (m + S(x))^2 - (E - qV(x))^2\right]\psi(x) \tag{23}$$

where the vector and the scalar potential are chosen linear as follows

$$\begin{aligned} qV(x) &= V_0 x \\ S(x) &= S_0 x. \end{aligned} \tag{24}$$

and we take $S_0^2 - V_0^2 > 0$ so as to avoid complex eigenvalues. We replace $S(x)$ and $V(x)$ and using the representation (5) and (24), the Eq. (23) becomes:

$$\left[\frac{d^2}{du^2} + \frac{2}{u}\frac{d}{du} + \frac{A}{u^2} + \frac{B}{u} - C^2\right]\psi(u) = 0 \tag{25}$$

where we have used the same transformation (13) and this notation

$$\begin{aligned} A &= \frac{V_0^2 - S_0^2}{\gamma^4} + \frac{2(EV_0 + mS_0)}{\gamma^3} + \frac{E^2 - m^2}{\gamma^2} + \frac{1}{4}, \\ B &= \frac{2(S_0^2 - V_0^2)}{\gamma^4} - \frac{2(EV_0 + mS_0)}{\gamma^3}, \\ C &= \frac{\sqrt{S_0^2 - V_0^2}}{\gamma^2}, \end{aligned} \tag{26}$$

To simplify the Eq. (25), we introduce,

$$\begin{aligned} \psi(u) &= u^\sigma \exp(-Cu)F(u), \\ u &\mapsto y = 2Cu, \end{aligned} \tag{27}$$

so, the differential equation will reduce to the equation of the associated Laguerre polynomials $L_n^k(y)$,

$$\left[y\frac{d^2}{dy^2} + [(2\sigma + 2) - y]\frac{d}{dy} + \frac{1}{y}[\sigma(\sigma - 1) + 2\sigma + A] + \frac{1}{2C}[B - 2C - 2C\sigma]\right]F(y) = 0. \tag{28}$$

by imposing the constraint,

$$\sigma(\sigma-1)+2\sigma+A=0, \tag{29}$$

to eliminate the coefficient proportional to $\frac{1}{y}$, and

$$\begin{cases} \frac{1}{2C}\left[B-2C-2C\sigma\right]=n, \\ 2\sigma+2=k+1. \end{cases} \tag{30}$$

The relation (29) leads to the following expressions for σ by

$$\sigma_{\pm}=-\frac{1}{2}\pm\frac{1}{\gamma}\sqrt{\left(m+E-\frac{(S_0-V_0)}{\gamma}\right)\left(m-E-\frac{(S_0+V_0)}{\gamma}\right)} \tag{31}$$

Among these two solutions, the physically acceptable one is only σ_+. To extract the energy spectrum, we substitute the expression (31) into the first relation of (30), then it is straightforward to show that

$$E^{\pm}=-\frac{mV_0}{S_0}-\gamma\frac{V_0\sqrt{S_0^2-V_0^2}}{S_0^2}\left(n+\frac{1}{2}\right)\pm\frac{S_0^2-V_0^2}{S_0^2}\sqrt{-\gamma^2\left(n+\frac{1}{2}\right)^2-\frac{\gamma\,(2n+1)\,mS_0}{\sqrt{S_0^2-V_0^2}}+\frac{(2n+1)\,S_0^2}{\sqrt{S_0^2-V_0^2}}}, \tag{32}$$

It is remarkable that the expression of the energy spectrum is a dependent function of the deformation parameter γ, γ^2 and with powers in n, n^2 which explains the phenomenon of confinement due to the new type of extended uncertainty principle. Moreover , for large values of n, the second term is not defined of $E^{\pm}$. In order to ensure the positivity of the square root of energy, one must impose an upper bound on the allowed values of n.

Solving the Eq. (23) along with (27), (28) and (31), we obtain the final form of the wave function in the former variable x as

$$\psi(x)=N_{n\lambda}\,(1+\gamma x)^{-\frac{1}{2}+\frac{1}{\gamma}\sqrt{\left(m+E-\frac{(S_0-V_0)}{\gamma}\right)\left(m-E-\frac{(S_0+V_0)}{\gamma}\right)}}\exp\left\{-\frac{\sqrt{(S_0^2-V_0^2)}}{\gamma^2}(1+\gamma x)\right\}$$
$$L_n^{\frac{2}{\gamma}\sqrt{\left(m+E-\frac{(S_0-V_0)}{\gamma}\right)\left(m-E-\frac{(S_0+V_0)}{\gamma}\right)}}\left(\frac{2\sqrt{(S_0^2-V_0^2)}}{\gamma^2}(1+\gamma x)\right), \tag{33}$$

and N_{nr} is a normalization constant.

Now if we consider $\gamma=0$ absence of deformation, we replace $\gamma=0$ in (32) we find,

$$E^{\pm}=-\frac{mV_0}{S_0}\pm\frac{\left(S_0^2-V_0^2\right)^{\frac{3}{4}}}{S_0}\sqrt{(2n+1)} \tag{34}$$

which is the result of the ordinary case [61,62].

4 The Klein Gordon Equation with Mixed Scalar and Vector Inversely Linear Potentials

In this case we choose the vector and the scalar potential inversely linear of Coulomb-type as follows

$$qV(x)=\frac{V_0}{\mid x\mid}$$
$$S(x)=\frac{S_0}{\mid x\mid}, \tag{35}$$

Using the transformation $u=1+\gamma\mid x\mid$ and the representation (5), for $x>0$, the stationary Klein–Gordon equation in (1 + 1) dimension in the framework of of new type of EUP (23) can be written as :

$$\left[\frac{d^2}{du^2}+\frac{2}{u}\frac{d}{du}+\frac{a_1}{u^2}+\frac{a_2}{u(1-u)}-\frac{a_3^2}{(1-u)^2}\right]\Psi(u)=0 \tag{36}$$

where we replaced $S(x)$ and $V(x)$ by their expressions (35) and this notation,

$$\begin{aligned} a_1 &= \frac{2(EV_0+mS_0)}{\gamma} - (S_0^2 - V_0^2) + \frac{(E^2-m^2)}{\gamma^2} + \frac{1}{4}, \\ a_2 &= \frac{2(EV_0+mS_0)}{\gamma} - 2(S_0^2 - V_0^2), \\ a_3 &= \sqrt{S_0^2 - V_0^2)}\,,\ S_0 > V_0 \end{aligned} \tag{37}$$

In addition, we note that this Eq. (36) possesses three singular points $0, 1, \infty$. By means of the substitution $\Psi(u) = u^p(1-u)^q\varphi(u)$, this equation will reduce to the hypergeometric type

$$\left[u(1-u)\frac{d^2}{du^2} + [(2p+2)-(2p+2q+2)u]\frac{d}{du} + [a_2 - 2pq - 2q]\right]\varphi(u) = 0. \tag{38}$$

where p and q are fixed as follows,

$$\begin{cases} p = -\frac{1}{2} \pm \sqrt{(S_0^2 - V_0^2) - \frac{2(EV_0+mS_0)}{\gamma} - \frac{(E^2-m^2)}{\gamma^2}} \\ q = \frac{1}{2} \pm \sqrt{\frac{1}{4} + (S_0^2 - V_0^2)}, \end{cases} \tag{39}$$

and the solution of Eq. (38) can be written as

$$\varphi(u) \sim {}_2F_1(a, b; c; u) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{u^k}{k!} \tag{40}$$

with the parameters a, b and c are given by

$$\begin{cases} a = p + q + \frac{1}{2} - i\sqrt{\frac{(E^2-m^2)}{\gamma^2}} \\ b = p + q + \frac{1}{2} + i\sqrt{\frac{(E^2-m^2)}{\gamma^2}} \\ c = 1 \pm 2\sqrt{(S_0^2 - V_0^2) - \frac{2(EV_0+mS_0)}{\gamma} - \frac{(E^2-m^2)}{\gamma^2}} \end{cases} \tag{41}$$

The mathematical solutions of Eq. (36) in the former variable x as

$$\Psi(x)) = N_\gamma(1+\gamma x)^p x^q {}_2F_1(a, b; c; 1+\gamma x)\,, \tag{42}$$

where N_γ is the normalization constant and the boundary condition that ($u \longrightarrow 1 \text{or} x \longrightarrow 0$) leads the wave function tending to finite, the hypergeometric function reduced to a polynomial with the following restriction

$$a = -n, \tag{43}$$

which is the quantization rule of the system and gives us the energy eigenvalues as

$$\begin{aligned} E_n^{\pm} &= -V_0\frac{mS_0 + \frac{\gamma}{2}\left[(n+q)^2 - (S_0^2 - V_0^2)\right]}{V_0^2 + (n+q)^2} \\ &\pm \frac{1}{2}\left\{\frac{V_0^2\left[2\gamma mS_0 + \gamma^2\left((n+q)^2 + (V_0^2 - S_0^2)\right)\right]^2}{\left[V_0^2 + (n+q)^2\right]^2}\right. \\ &\left. + \frac{4m^2\left[(n+q)^2 - S_0^2\right] - 4\gamma mS_0\left[(n+q)^2 - (S_0^2 - V_0^2)\right] - \gamma^2\left[(n+q)^2 - (S_0^2 - V_0^2)\right]^2}{V_0^2 + (n+q)^2}\right\}^{\frac{1}{2}} \end{aligned} \tag{44}$$

Also for this case, for large values of n, the second term is not defined . In order to ensure the positivity of the square root of energy, one must impose an upper bound on the allowed values of n.

Now in our analysis, it is interesting to study two particular cases

First; if $\gamma = 0$ absence of deformation, we replace $\gamma = 0$ in (44) we find,

$$E_n^{\pm} = \frac{-mV_0S_0}{V_0^2 + (n+q)^2} \pm m\sqrt{\frac{(n+q)^2 - S_0^2}{V_0^2 + (n+q)^2}} \tag{45}$$

Second, if $\gamma = 0$ and $S_0 = 0$, taking $(\gamma \to 0)$ and $S_0 = 0$, the expression of energy spectrum (44) become

$$E_n^{\pm} = \pm \frac{m}{\sqrt{1 + \frac{V_0^2}{(n+q)^2}}} \tag{46}$$

which coincides exactly with those of the literatures [63].

At the end of this section, we mention that in the region $x < 0$, we get the same form of the solution (42) if we make the change of the variables $y = -x$.

5 Conclusion

In this contribution, we have established an exact and explicit solution of some problems in the context of new type of the extended uncertainty principle using the displacement operator method such as: The Klein–Gordon particle confined in a one dimensional box, the scalar particle with linear vector and scalar potentials and the case of Coulomb-type vector and scalar potentials. In these three cases, the exact analytical solution is determined, the wave functions and the exact energy spectrum are obtained depending on the deformation parameter γ. On the other hand, the expressions of energy spectrum vary with all the power of n, which explain the confinement phenomenon. Also, it is mentioned that for the last two cases, bound states are limited, the expressions of energy are not defined for large values of n, one must impose an upper bound on the allowed values of n. Finally the limiting cases are presented.

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The EUP Dirac Oscillator: A Path Integral Approach

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Abstract The Green function for Dirac oscillator in (1+1) dimension in the context of the extended uncertainty principle (EUP) is calculated exactly via the path integral formalism. The spectrum energy is determined, the corresponding wave functions suitably normalized are derived and they are expressed by the Gegenbauer's polynomials. Special cases are considered.

1 Introduction

It is well known that the spin is a fundamental physical quantity in quantum physics and plays a significant role in various areas of physics, in particular, in the explanation of the mesoscopic phenomena. In the relativistic case, the exact analytical solutions of physical models much required, enables us to explore at the same time relativistic and spin effects, and the relativistic principles require that space-time must be described in the unified manner. Indeed, Feynman's path integral formulation for systems with spin has not yet been definitively achieved due to the discrete nature of the spin and the requirements of relativistic invariance. In fact, path integral uses classical and continuous concepts such as trajectories whereas the spin is irreducibly of a discrete nature, without classical equivalent, and to satisfy the relativistic invariance requirements on the other hand. To overcome this difficulty within this framework, some models were presented for this purpose. For example, the Feynman attempt for the free Dirac electron using the Poisson stochastic process [1], the Schulman description of the spin of a nonrelativistic particle by the top model using the three Euler angles [2], and its extension to the relativistic case [3], the Barut–Zanghi theory for the classical spinning electron related to zitterbewegung [4], the bosonic and fermionic Schwinger model in the related coherent state space [5–7] and the supersymmetric model using the Grassmann variables for the spin evolution with many developments [8–11].

Recently, the applicability of this Feynman formulation for the spin system has undergone notable development in various domains of physics with different topologies modeled by deformed algebras. For example, effects of the gravitational field in quantum mechanics in presence of the generalized uncertainty principle (GUP) [12–14], and on the noncommutative geometry in quantum system [15,16]. Consequently, in this regard, a significant number of papers have been published. Citing for instance, within the GUP framework the

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spinning particle subjected to the action of combined vector and scalar potentials [17,18] and the Dirac oscillator [19,20]. In noncommutative space, the Klein–Gordon and Dirac oscillators [21], and the harmonic oscillator related to energy-dependent potential [22]. And others important similar references using path approach as, the Klein–Gordon equation with the energy dependent linear and Coulomb potentials is treated in [23] and the harmonic oscillator and the radial hydrogen atom propagators related to energy-dependent potentials are analyzed [24].

Furthermore, there another type of deformation due to the topology of the physical space called the extended uncertainty principle (EUP) [25–30]. As an example in the (anti)-de sitter background, the Heisenberg uncertainty principle is modified by introducing corrections proportional to the cosmological constant $\Lambda = -3\lambda^2$, where $\lambda^2 < 0$ indicates de-Sitter space-time, and $\lambda^2 > 0$ for the anti-de Sitter space-time. This latter is invariant with special relativity (SR).

Our attempt through this work is to set up a path integral formulation to establish the Green function for the Dirac oscillator problem in the context of this EUP. At the same time, it is important to remember that the Dirac oscillator was introduced for the first times by [31,32]. It was the subject of many developments and received considerable study in various areas of physics. For example, it appears in quantum optics [33], in nuclear physics [34], in noncommutative geometry [35] and in graphene physics [36]. It is used as the confining part of the phenomenological Cornell potential and an intergroup potential in quantum chromodynamics.

2 Quantum Mechanics on (Anti)-de Sitter Background

The EUP can be obtained from the definition of quantum mechanics on (anti)-de sitter space-time. It is well known that (anti)-de Sitter space-time can be realized as a hyperboloid of equation $\eta_{ab}\zeta^a\zeta^b = \pm R^2$ embedded in five-dimensional Minkowski space, with coordinates $\zeta^a (a = 0, 1, 2, 3, 4)$ and corresponding metric is $\eta_{ab} = diag\,(1, -1, -1, -1, \pm 1)$. That will be reduced to ordinary special relativity when $R \to \infty$ [37],

$$ds^2 = \eta_{ab} d\zeta^a d\zeta^b = B_{\mu\nu}(x)\, dx^\mu dx^\nu; \quad \mu, \nu = 0, 1, 2, 3, \tag{1}$$

where the parametrization of the hyperboloid is given by projective (Beltrami) coordinates [38],

$$x_\mu = \frac{\zeta_\mu}{\zeta_4}, \; B^{\mu\nu}(x) = \left(1 - \frac{\eta_{\sigma\tau} x^\sigma x^\tau}{R^2}\right)\left(\eta^{\mu\nu} - \frac{x^\mu x^\nu}{R^2}\right). \tag{2}$$

$B^{\mu\nu}(x)$ is Beltrami metric, it is similar to the Minkowski one in flat space-time and the Beltrami de sitter ($\mathcal{B}dS$) space-time is the dS space-time with Beltrami metric. The generators of de-Sitter in Beltrami coordinates and the momentum operators satisfy the following commutation relations [25,39,40], (throughout this paper we adopt the natural units $\hbar = c = 1$)

$$\left[\hat{J}_{\mu\nu}, \hat{J}_{\sigma\rho}\right] = i\left(\eta_{\nu\rho}\hat{J}_{\mu\sigma} - \eta_{\nu\sigma}\hat{J}_{\mu\rho} + \eta_{\mu\sigma}\hat{J}_{\nu\rho} - \eta_{\mu\rho}\hat{J}_{\nu\sigma}\right), \tag{3}$$

$$\left[\hat{J}_{\mu\nu}, \hat{p}_\rho\right] = i\left(\eta_{\mu\rho}\hat{p}_\nu - \eta_{\nu\rho}\hat{p}_\mu\right), \quad \left[\hat{p}_\mu, \hat{p}_\nu\right] = \frac{i\hat{J}_{\mu\nu}}{R^2}, \tag{4}$$

$$\left[\hat{x}_\mu, \hat{p}_\nu\right] = i\left(\eta_{\mu\nu} + \frac{\hat{x}_\mu\hat{x}_\nu}{R^2}\right), \; \left[\hat{x}_\mu, \hat{x}_\nu\right] = 0 \quad \text{and } \mu, \nu = 0, 1, 2, 3, \tag{5}$$

where $\hat{J}_{\mu\nu}$ are the generators of Lorentz transformations given as $\hat{J}_{\mu\nu} = \hat{x}_\mu\hat{p}_\nu - \hat{x}_\nu\hat{p}_\mu$.

For the anti-de-Sitter space and in the case of the one-dimensional space, the modified commutation relations leading to the extended commutation relations is given as [41],

$$\left[\hat{X}, \hat{P}\right] = i\left(1 + \alpha\hat{X}^2\right), \tag{6}$$

where α is a positive deformation parameter proportional to the cosmological constant, or inversely proportional to the square of the anti-de Sitter radius ($\alpha = H^2$: H^2 is the Hubble rate). Which lead to the following EUP:

$$(\Delta X)(\Delta P) \geqslant \frac{1}{2}\left(1 + \alpha(\Delta X)^2\right), \tag{7}$$

which indicates the emergence of a nonzero minimal uncertainty in momentum. The minimization of (7) with respect to ΔX gives

$$(\Delta P)_{\min} = \frac{\sqrt{\alpha}}{2}. \tag{8}$$

with α is positive.

According to (6), the $\hat{X}$ and $\hat{P}$ operators in this representation can be realized by operators $\hat{x}$ and $\hat{p}$, as follows:

$$\begin{cases} \hat{X} = \hat{x} \\ \hat{P} = \left(1 + \alpha x^2\right) \hat{p} \end{cases}, \tag{9}$$

with $\hat{x}$ and $\hat{p}$ satisfy the usual Heisenberg canonical commutation relation: $\left[\hat{x}, \hat{p}\right] = i$. We note that the momentum operator $\hat{P}$ is not symmetric in all Hilbert space $L^2\left(\mathbb{R}, dx\right)$. For this, we need to change this space into subspace $L^2\left(\mathbb{R},\ d_\alpha x = \frac{dx}{1+\alpha x^2}\right)$. This makes the modified scalar product of two functions $\psi(x)$ and $\varphi(x)$ in position space basis $\{|x\rangle\}$ as

$$\langle \varphi \mid \psi \rangle = \int \varphi^* (x)\, \psi (x)\, d_\alpha x. \tag{10}$$

From this modification, we can construct the closure relation as follows

$$\int_{-\infty}^{+\infty} d_\alpha x \, |x\rangle \langle x| = 1, \tag{11}$$

and the corresponding projection relation is

$$\langle x | x' \rangle = \left(1 + \alpha x^2\right) \delta\left(x - x'\right), \tag{12}$$

otherwise

$$\langle x | x' \rangle = \int_{-\infty}^{+\infty} \frac{dp}{2\pi} \exp\left[\frac{\imath p}{\sqrt{\alpha}} \left(\arctan \sqrt{\alpha} x - \arctan \sqrt{\alpha} x'\right)\right]. \tag{13}$$

Then we use the simplified form

$$\left\langle x \left| \hat{P} \right| x' \right\rangle = -\int_{-\infty}^{+\infty} \frac{p dp}{2\pi} \exp\left[\frac{\imath p}{\sqrt{\alpha}} \left(\arctan \sqrt{\alpha} x - \arctan \sqrt{\alpha} x'\right)\right]. \tag{14}$$

Assuming no deformation for the time component, we have

$$\langle x_0 | x_0' \rangle = \delta\left(x_0 - x_0'\right), \quad \int_{-\infty}^{+\infty} dx_0 \, |x_0\rangle \langle x_0| = 1. \tag{15}$$

Moreover, for the de-Sitter space, it can be constructed by replacing $\alpha \longrightarrow (-\alpha')$ where α' is a positive parameter, we have

$$(\Delta X)(\Delta P) \geq \frac{1}{2}\left(1 - \alpha' (\Delta X)^2\right). \tag{16}$$

This latter leads to no minimum momentum uncertainty, as we write

$$-\frac{(\Delta P)}{\alpha'} - \frac{1}{\alpha'}\sqrt{\alpha' + (\Delta P)^2} \leq (\Delta X) \leq -\frac{(\Delta P)}{\alpha'} + \frac{1}{\alpha'}\sqrt{\alpha' + (\Delta P)^2}. \tag{17}$$

Then Eq. (17) it becomes bounded $-\frac{1}{\sqrt{\alpha'}} \leq (\Delta X) \leq \frac{1}{\sqrt{\alpha'}}$ in the limit $(\Delta P_i) \rightarrow 0$. While the representations of $\hat{X}$ and $\hat{P}$ can be thought of as,

$$\hat{X} = x; \ \ \hat{P} = -\imath \left(1 - \alpha' x^2\right) \frac{\partial}{\partial x}. \tag{18}$$

In the following section, we concentrate on the explicit calculation of the Green function for relativistic Dirac oscillators in the context of the EUP, by using the path integral formalism.

3 Derivation of Path Integral for the 1D-Dirac Oscillator

As it is known, the Dirac oscillator propagator in $(1+1)$ dimensions is the causal Green function $S^{(c)}(x_b, x_a)$ of the Dirac oscillator equation, which is defined as

$$\left(\gamma^{\mu}\hat{\Pi}_{b\mu} - m\right) S^{(AdS)}\left(x_b^{\mu}, x_a^{\mu}\right) = -\left(1+\alpha x_b^2\right)\delta\left(x_b - x_a\right)\delta\left(t_b - t_a\right), \tag{19}$$

where the components of $\hat{\Pi}_{\mu}$ are expressed as

$$\hat{\Pi}_0 = \hat{P}_0, \;\; \hat{\Pi}_1 = \hat{P} - \imath m\omega\gamma^0\hat{X}. \tag{20}$$

Here the operators $(\hat{X}, \hat{P})$ satisfy the commutation relations of the EUP, which is defined in the relation (6). While γ_{μ} are the Dirac matrices verify the commutation relation $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu}$, with the metric $\eta^{\mu\nu} = \text{diag}(1, -1)$ and $\mu, \nu = 0, 1$. These Dirac matrices can be chosen in terms of Pauli matrices σ^i as follows:

$$\gamma^1 = i\sigma^1 \quad \text{and} \quad \gamma^0 = \sigma^3. \tag{21}$$

The corresponding solution of Eq. (19) is defined as the inverse of the Dirac operator $\mathcal{O}_-^d$,

$$S^{(AdS)}(x_b, x_a) = -\left\langle x_b \left| \left[\mathcal{O}_-^d\right]^{-1} \right| x_a \right\rangle = -\left\langle x_b \left| \mathcal{O}_+^d \left[\mathcal{O}_-^d \mathcal{O}_+^d\right]^{-1} \right| x_a \right\rangle. \tag{22}$$

$\mathcal{O}_{\pm}^d$ are defined as $\gamma^{\mu}\hat{\Pi}_{\mu} \pm m$ and $\mathcal{O}_+^d$ represents to the global Dirac projection operator. Now, following [42], the global representation for the causal Green function is obtained by inserting the completeness relation for the space-time states given by Eqs. (11) and (15) between the operators $\mathcal{O}_+^d$ and $\left[\mathcal{O}_-^d \mathcal{O}_+^d\right]^{-1}$, we get

$$S^{(AdS)}(x_b, x_a) = \left(\gamma^{\mu}\hat{\Pi}_{\mu} + m\right)_b G^{(AdS)}(x_b, x_a). \tag{23}$$

Here $G^{(AdS)}(x_b, x_a)$ is the global Green function defined as

$$G^{(AdS)}(x_b, x_a) = -\left\langle x_b \left| \left[\mathcal{O}_-^d \mathcal{O}_+^d\right]^{-1} \right| x_a \right\rangle, \tag{24}$$

and introducing the Schwinger proper-time method, the Green function $G^{(g)}\left(x_b^{\mu}, x_a^{\mu}\right)$ becomes

$$G^{(AdS)}\left(x_b^{\mu}, x_a^{\mu}\right) = \imath \int_0^{+\infty} d\lambda \left\langle x_b^{\mu} \left| \exp\left(\imath\lambda\mathcal{H}\right) \right| x_a^{\mu} \right\rangle. \tag{25}$$

$\mathcal{H}$ is the Hamiltonian of the system in question whose quadratic form, and will reduce to

$$\mathcal{H} = \mathcal{O}_-^d \mathcal{O}_+^d = \hat{P}_0^2 - \hat{P}^2 - m^2\omega^2\hat{X}^2 - m^2 + m\omega\gamma^0\left(1 + \alpha\hat{X}^2\right), \tag{26}$$

which is associated the case of anti-de Sitter space and their corresponding representation (9). Then by taking into account the properties of the following exponential matrix, will simplify to:

$$\exp\left[\imath\lambda m\omega\gamma^0\left(1+\alpha\hat{X}^2\right)\right] = \cos\left(\lambda m\omega\left(1+\alpha\hat{X}^2\right)\right) + \imath\gamma^0\sin\left(\lambda m\omega\left(1+\alpha\hat{X}^2\right)\right), \tag{27}$$

or in another form

$$\exp\left[\imath\lambda m\omega\gamma^0\left(1+\alpha\hat{X}^2\right)\right] = \sum_{s=\pm1}\chi_s\chi_s^{\dagger}\exp\left[\imath\lambda s m\omega\left(1+\alpha\hat{X}^2\right)\right], \tag{28}$$

where $\chi_s^{\dagger} = \frac{1}{2}\left(1+s,\; 1-s\right)$. Substituting (28) into (25), the global Green function can write as follow

$$G^{(AdS)}\left(x_b^{\mu}, x_a^{\mu}\right) = \sum_{s=\pm1}\chi_s\chi_s^{\dagger}\mathcal{G}^{(AdS)}\left(x_b^{\mu}, x_a^{\mu}\right). \tag{29}$$

$\mathcal{G}^{(dS)}\left(x_b^\mu, x_a^\mu\right)$ is the new global representation defined by:

$$\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right) = \imath \int_0^{+\infty} d\lambda \left\langle x_b^\mu \left| \exp\left(-\imath\lambda\mathcal{H}^{(s)}\right) \right| x_a^\mu \right\rangle, \tag{30}$$

and

$$\mathcal{H}^{(s)} = -\left[\hat{P}_0^2 - \hat{P}^2 - m^2 - (m\omega)^2 \hat{X}^2 + sm\omega\left(1+\alpha\hat{X}^2\right)\right]. \tag{31}$$

At this level, to derive a path integral representation for $\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right)$, we follow the standard discretization method, we write $\exp\left(-\imath\lambda\mathcal{H}^{(s)}\right)$ by $\left[\exp\left(-\imath\lambda\mathcal{H}^{(s)}/(N+1)\right)\right]^{N+1}$, and we insert N times the identities of Eqs. (11) and (15) between each pair of operators $\exp\left(-\imath\varepsilon\lambda\mathcal{H}^{(s)}\right)$ infinitesimal with $\varepsilon = 1/(N+1)$. Then, taking at the end, the limit $N \to \infty$, the expression of $\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right)$ becomes as,

$$\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right) = -\imath \lim_{N\to\infty} \int_0^{\infty} d\lambda \prod_{j=1}^{N} \left[\int d_\alpha x_j dx_{0j}\right] \prod_{j=1}^{N+1} \left\langle x_j, x_{0j} \right| \left[1 - \imath\varepsilon\lambda\mathcal{H}^{(s)} + O\left(\varepsilon^{n\geqslant 2}\right)\right] \left| x_{j-1} x_{0j-1} \right\rangle, \tag{32}$$

where $x_0 = x_a, x_{00} = x_{0a}, x_{N+1} = x_b$ and $x_{0N+1} = x_{0b}$. Using the relations (9) and (14) into (32), and introducing the integral representation follow:

$$\left\langle x_j, x_{0j} \mid x_{j-1}, x_{0j-1} \right\rangle = \int \frac{dp_{0j}}{2\pi} \exp\left(\imath p_{0j}\Delta x_{0j}\right) \int \frac{dp_j}{2\pi} \exp\left(\imath p_j \frac{\Delta \arctan(\sqrt{\alpha}x_j)}{\sqrt{\alpha}}\right), \tag{33}$$

we will get the following expression

$$\begin{aligned}\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right) = -\imath \lim_{N\to\infty} \int_0^{\infty} d\lambda \prod_{j=1}^{N} \left[\int d_\alpha x_j dx_{0j}\right] \prod_{j=1}^{N+1} \left[\int \frac{dp_j}{2\pi}\frac{dp_{0j}}{2\pi}\right] \\ \times \exp\left\{\imath \textstyle\sum_{j=1}^{N+1} \left[p_j \frac{\Delta \arctan(\sqrt{\alpha}x_j)}{\sqrt{\alpha}} + p_{0j}\Delta x_{0j} + \varepsilon\lambda\left(p_{0j}^2\right.\right.\right. \\ \left.\left.\left. -p_j^2 - (m\omega)^2 x_j^2 - m^2 + sm\omega\left(1+\alpha x_j^2\right)\right)\right]\right\}.\end{aligned} \tag{34}$$

The Gaussian integration on the p_j, x_{0j} and p_{0j} variables is immediate, we obtain

$$\begin{aligned}\mathcal{G}^{(AdS)}\left(x_b^\mu, x_a^\mu\right) = -\imath \lim_{N\to\infty} \int_0^{\infty} d\lambda \int \frac{dp_0}{2\pi} e^{ip_0(x_{0b}-x_{0a})} \prod_{j=1}^{N} \int d_\alpha x_j \prod_{j=1}^{N+1} \frac{1}{\sqrt{4\pi\imath\lambda\varepsilon}} \\ \times \exp\left\{\imath \textstyle\sum_{j=1}^{N+1} \left[\frac{\left(\Delta \arctan(\sqrt{\alpha}x_j)\right)^2}{4\lambda\varepsilon\alpha} + \varepsilon\lambda\left(p_0^2 - (m\omega)^2 x_j^2 - m^2 + sm\omega\left(1+\alpha x_j^2\right)\right)\right]\right\}.\end{aligned} \tag{35}$$

It is remarkable that our system converted to the case of the position-dependent effective mass. Now, in order to make this expression to the ordinary form of the Feynman path integral with constant mass, we will use the following coordinate transformation method,

$$y = f(x), \quad y_0 = x_0. \tag{36}$$

This new $y-$variable changes in the interval $\left]-\frac{\pi}{2\sqrt{\alpha}}, +\frac{\pi}{2\sqrt{\alpha}}\right[$ according to variables of x in the interval $]-\infty, +\infty[$. In order to determine all quantum fluctuations, we perform the corrections associated with measure and action terms: $\left(dx_j/(1+\alpha x_j^2)\right)$ and $\left(\left(\Delta\arctan\left(\sqrt{\alpha}x_j\right)\right)^2/4\lambda\varepsilon\alpha\right)$ to get the conventional form of Feynman path integral. To determine the appropriate corrections and avoid any ambiguities, we discretize the measure and choose for any δ-point discretization interval $(x_J^{(\delta)} = \delta x_J + (1-\delta)\, x_{J-1})$ according to the technique used in [19]. So after straightforward calculations, we obtain the total quantum correction with two approaches, Kleinert method [43] and standard method [44]

$$C^{T}_{Kleinert} = 2i\varepsilon\lambda \frac{m^2\omega^2}{\alpha}\left(1+\tan^2(\sqrt{\alpha}y)\right)\delta\left(2\delta-1\right), \tag{37}$$

and

$$C^{T}_{Khand} = 2i\varepsilon\lambda \frac{m^2\omega^2}{\alpha}\left[\left(1-8\delta+8\delta^2\right)\tan^2(\sqrt{\alpha}y)\right]. \tag{38}$$

In order to obtain the exact results we should give the two values of δ, when using Kleinert method [43] we will find $\delta = 0, 1/2$, and when using a standard method [44] we get $\delta = \frac{1}{2}\left(1 \pm 1/\sqrt{2}\right)$. These points are the same obtained in one dimension case with generalized uncertainty principle [19]. Under these considerations we will simplify C_T to zero, and the amplitude $\mathcal{G}^{(AdS)}\left(y_b^{\mu}, y_a^{\mu}\right)$ becomes as follows:

$$\mathcal{G}^{(AdS)}\left(y_b^{\mu}, y_a^{\mu}\right) = -\imath\int_0^{\infty} d\lambda \prod_{j=1}^{N}\int \frac{dp_0}{2\pi}e^{ip_0(y_{0b}-y_{0a})}\prod_{j=1}^{N+1}e^{\imath\lambda\left[p_0^2-m^2+sm\omega\right]} \times \mathcal{K}^{(PT)}\left(y_b^{\mu}, y_a^{\mu}\right). \tag{39}$$

Here $\mathcal{K}^{(PT)}\left(y_b^{\mu}, y_a^{\mu}\right)$ is identical the propagator to the standard problem of the Poschl–Teller potential defined by

$$\mathcal{K}^{(PT)}\left(y_b^{\mu}, y_a^{\mu}\right) = \lim_{N\to\infty}\prod_{j=1}^{N}\int dy_j \prod_{j=1}^{N+1}\frac{1}{\sqrt{4\pi\imath\lambda\varepsilon}}\exp\left\{\imath\left[\frac{\left(\Delta y_j\right)^2}{4\lambda\varepsilon} - \varepsilon\lambda\left(\frac{m^2\omega^2}{\alpha} - sm\omega\right)\tan^2\left(\sqrt{\alpha}y_j\right)\right]\right\}. \tag{40}$$

While in the case of de-Sitter space we replace α by $\left(-\alpha'\right)$, and it is given the standard problem of the modified Poschl–Teller potential.

4 Calculation of the Propagator

The path integral of $y\left(t_j\right)$ in Eq. (40) (i.e., anti-de Sitter space) is exactly the propagator associated with the Poschl–Teller potential. Which is solved exactly in Refs. [44,45], and equals

$$\mathcal{G}^{(AdS)}\left(y_b^{\mu}, y_a^{\mu}\right) = -\imath\lim_{N\to\infty}\int_0^{\infty} d\lambda \int \frac{dp_0}{2\pi}e^{ip_0(y_{0b}-y_{0a})}\, e^{\imath\lambda\left[p_0^2-m^2+sm\omega\right]} \times \left[\sum_{n=0}^{\infty} e^{-\imath\lambda E_{n,s}^{(PT)}}\Psi_{n,s}^{(PT)}\left(y_b\right)\left(\Psi_{n,s}^{(PT)}\right)^{*}\left(y_a\right)\right], \tag{41}$$

with $E_{n,s}^{(PT)}$ is the energy spectrum associated to the Poschl–Teller potential, which is defined as

$$E_{n,s}^{(PT)} = \alpha\left(n^2 + (2n+1)\,\eta_s\right), \tag{42}$$

and $\psi_{n,s}^{(PT)}(y)$ are corresponding the wave functions and given by

$$\psi_{n,s}^{(PT)}(y) = \Gamma\left(\eta_s\right)\sqrt{\frac{2^{2\eta_s-1}n!\left(n+\eta_s\right)\sqrt{\alpha}}{\pi\Gamma\left(n+2\eta_s\right)}}\left(\cos\left(\sqrt{\alpha}y\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y\right)\right), \tag{43}$$

where

$$\eta_s = \frac{1-s}{2} + \frac{m\omega}{\alpha}. \tag{44}$$

η_s is parameter determined according the condition of the extended uncertainty principle, and the characteristic length of oscillator, retaining the solution associated with $\frac{m\omega}{\alpha} > 1$, the values $\eta_+ = \frac{m\omega}{\alpha}$, $\eta_- = 1 + \frac{m\omega}{\alpha}$ are accepted. While the other negatives values are rejected. Substituting (41) into (29), the propagator becomes as:

$$G^{(AdS)}\left(y_b^{\mu}, y_a^{\mu}\right) = -\imath \int_0^{\infty} d\lambda \int \frac{dp_0}{2\pi} e^{ip_0(y_{0b}-y_{0a})} \sum_n \sum_{s=\pm 1} \chi_s \chi_s^{\dagger} e^{\imath\lambda\left[p_0^2 - m^2 + sm\omega - E_{n,s}^{(PT)}\right]}$$
$$\times \left[(\Gamma(\eta_s))^2 \frac{2^{2\eta_s-1} n!\,(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)} \left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\right.$$
$$\left.\times \left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right]. \tag{45}$$

An integration over λ to give this expression

$$G^{(AdS)}\left(y_b^{\mu}, y_a^{\mu}\right) = -\imath \sum_n \sum_{s=\pm 1} (\Gamma(\eta_s))^2 \frac{2^{2\eta_s-1} n!\,(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)} \int \frac{dp_0}{2\pi} \frac{e^{ip_0(y_{0b}-y_{0a})}}{p_0^2 - \mathcal{E}_{n,s}} \chi_s \chi_s^{\dagger}$$
$$\times \left[\left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right], \tag{46}$$

with

$$\mathcal{E}_{n,s}(p_0) = m^2 - sm\omega + E_{n,s}^{(PT)}. \tag{47}$$

The above Eq. (46) lacks the integration over energy p_0. This can be converted to a complex integration along the special contour C, and then using the residue theorem, we have:

$$\oint \frac{dp_0}{2\pi i} \frac{e^{ip_0(y_{0b}-y_{0a})}}{p_0^2 - \mathcal{E}_{n,s}} f(p_0) = \sum_{k=1}^{n} Res\left(\frac{e^{-iE(t_b-t_a)}}{E^2-\mathcal{E}_{n,s}} f(E), E_k\right) e^{-iE_k(y_{0b}-y_{0a})}$$
$$= \sum_{\epsilon=\pm 1} \frac{f\left(E_{n,s}^{(\epsilon)}\right)}{2\epsilon\omega_{n,s}^{(AdS)}} e^{-iE_{n,s}^{(\epsilon)}(y_{0b}-y_{0a})} \Theta\left(\epsilon\left(y_{0b}-y_{0a}\right)\right), \tag{48}$$

where $\epsilon = \pm 1$ and $\Theta(x)$ is the Heaviside function. This gives the following poles:

$$E_{n,s}^{(\epsilon)} = \epsilon\omega_{n,s}^{(AdS)} = \pm\sqrt{m^2 - sm\omega + \alpha\left(n^2 + (2n+1)\eta_s\right)}. \tag{49}$$

Using the residue theorem on global Green function expression defined in Eq. (46). The integrations over p_0 are carried, and becomes as

$$G^{(AdS)}\left(y_b^{\mu}, y_a^{\mu}\right) = -\imath \sum_{\epsilon=\pm 1} \sum_{s=\pm 1} \sum_n (\Gamma(\eta_s))^2 \frac{2^{2\eta_s-1} n!\,(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}$$
$$\times \left\{\frac{e^{-i\epsilon\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\epsilon\omega_{n,s}^{(AdS)}} \Theta\left(\epsilon\left(y_{0b}-y_{0a}\right)\right) \chi_s \chi_s^{\dagger} \left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\right.$$
$$\times \left[\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s} C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right]. \tag{50}$$

Furthermore, we can get the global Green function $G^{(dS)}\left(y_b^{\mu}, y_a^{\mu}\right)$ in de-Sitter Snyder case, just by changing α by $(-\alpha)$ with retaining the term of η_s.

5 Spectral Energies and Spinorial Wave Functions

To obtain the exact solutions for the wave functions and spectral energies for the system governed by the Dirac equation, it must bring the corresponding spectral decomposition of Dirac oscillator in $(1+1)$ dimension in the context of the EUP by the act operator $(\gamma^{\nu}\hat{\Pi}_{\nu} + m)_b$ on Eq. (50). This will be simplified as

$$S^{(AdS)}(x_b,x_a)=-\imath\left[\imath\sigma^3\frac{\partial}{\partial y_{0b}}+\sigma^1\left(\frac{\partial}{\partial y_b}+\sigma^3\frac{m\omega}{\sqrt{\alpha}}\tan\left(\sqrt{\alpha}y_b\right)\right)+m\right]$$
$$\times\sum_{s=\pm1}\sum_n\left\{(\Gamma(\eta_s))^2\frac{2^{2\eta_s-1}n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}\frac{e^{-is\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\omega_{n,s}^{(AdS)}}\Theta\left(s\left(y_{0b}-y_{0a}\right)\right)\chi_s\chi_s^\dagger\right.$$
$$\left.\times\left[\left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right]\right\}$$
$$+\left\{-(\Gamma(\eta_s))^2\frac{2^{2\eta_s-1}n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}\frac{e^{+is\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\omega_{n,s}^{(AdS)}}\Theta\left(-s\left(\left(y_{0b}-y_{0a}\right)\right)\right)\chi_s\chi_s^\dagger\right.$$
$$\left.\times\left[\left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_b\right)\right)\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s}C_n^{\eta_s}\left(\sin\left(\sqrt{\alpha}y_a\right)\right)\right]\right\}. \tag{51}$$

With some known relationships in algebra matrices for Dirac, we have

$$\sigma^3\chi_s=s\chi_s,\ \sigma^1\chi_s=\chi_{-s}\quad\text{and}\quad\sigma^2\chi_s=is\chi_{-s}, \tag{52}$$

and with helping of Gegenbauer's polynomials properties [46],

$$\begin{cases}\frac{d}{du}C_n^\eta(u)=2\eta C_{n-1}^{\eta+1}(u),\\ nC_n^\eta(u)=(2\eta+n-1)uC_{n-1}^\eta(u)-2\eta(1-u^2)C_{n-2}^{\eta+1}(u),\\ (2\eta+n)C_n^\eta(u)=2\eta\left[C_n^{\eta+1}(u)-uC_{n-1}^{\eta+1}(u)\right],\end{cases} \tag{53}$$

we can write the Green function through a straightforward calculation, as follows:

$$S^{(AdS)}(x_b,x_a)=-\imath\sum_{s=\pm1}\sum_n(\Gamma(\eta_s))^2\frac{2^{2\eta_s-1}n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}\left(\cos\left(\sqrt{\alpha}y_b\right)\right)^{\eta_s}\left(\cos\left(\sqrt{\alpha}y_a\right)\right)^{\eta_s}$$
$$\times\left\{\frac{e^{-is\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\omega_{n,s}^{(AdS)}}\Theta\left(s\left(y_{0b}-y_{0a}\right)\right)\left[\left(\omega_{n,s}^{(dS)}+m\right)C_n^{\eta_s}(\xi_b)C_n^{\eta_s}(\xi_a)\chi_s\chi_s^\dagger\right.\right.$$
$$\left.\left.+\sqrt{\alpha}\left[-\left(\frac{1-s}{2}\right)\left(1+\frac{m\omega}{\alpha}\right)\tan\left(\sqrt{\alpha}y_b\right)C_n^{\eta_s}(\xi_b)+2\eta_s\cos\left(\sqrt{\alpha}y_b\right)C_{n-1}^{\eta_s+1}(\xi_b)\right]C_n^{\eta_s}(\xi_a)\chi_{-s}\chi_s^\dagger\right]\right\}$$
$$-\left\{\frac{e^{+is\omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})}}{2\omega_{n,s}^{(AdS)}}\Theta\left(-s\left(y_{0b}-y_{0a}\right)\right)\left[\left(-\omega_{n,s}^{(dS)}+m\right)C_n^{\eta_s}(\xi_b)C_n^{\eta_s}(\xi_a)\chi_s\chi_s^\dagger\right.\right.$$
$$\left.\left.+\sqrt{\alpha}\left[-\left(\frac{1-s}{2}\right)\left(1+\frac{m\omega}{\alpha}\right)\tan\left(\sqrt{\alpha}y_b\right)C_n^{\eta_s}(\xi_b)+2\eta_s\cos\left(\sqrt{\alpha}y_b\right)C_{n-1}^{\eta_s+1}(\xi_b)\right]C_n^{\eta_s}(\xi_a)\chi_{-s}\chi_s^\dagger\right]\right\}. \tag{54}$$

Now, to obtain the spectral energies and corresponding eigenfunctions, we must unify the expression of energy $E_{n,s}^{(\epsilon)}$. Which leads us to make the following changes on the second term in the Green function, which are multiplied by $\Theta\left(-s\left(y_{0b}-y_{0a}\right)\right)$

$$\begin{aligned}s&\to s'=-s,\\ n&\to n'=n-s,\\ \eta_s&\to\eta_{s'}=\eta_s+s.\end{aligned} \tag{55}$$

After these changes, we can write

$$
\begin{aligned}
S^{(AdS)}(x_b, x_a) = -\imath \sum_{s=\pm 1} \sum_n (\Gamma(\eta_s))^2 \frac{2^{2\eta_s - 1} n! (n+\eta_s)\sqrt{\alpha}}{\pi \Gamma(n+2\eta_s)} \\
\times \left(\cos\left(\sqrt{\alpha} y_b\right)\right)^{\eta_s} \left(\cos\left(\sqrt{\alpha} y_a\right)\right)^{\eta_s} \frac{e^{-\imath s \omega_{n,s}^{(dS)}(y_{0b}-y_{0a})}}{2\omega_{n,s}^{(dS)}} \Theta\left(s\left(y_{0b}-y_{0a}\right)\right) \\
\times \left\{ \left[\left(\omega_{n,s}^{(dS)} + m\right) C_n^{\eta_s}(\xi_b) C_n^{\eta_s}(\xi_a) \chi_s \chi_s^\dagger \right.\right. \\
+ \sqrt{\alpha} \left[-\left(\frac{1-s}{2}\right)\left(1 + \frac{m\omega}{\alpha}\right) \tan\left(\sqrt{\alpha} y_b\right) C_n^{\eta_s}(\xi_b) \right. \\
\left.\left. +2\left(\frac{1-s}{2} + \frac{m\omega}{\alpha}\right) \cos\left(\sqrt{\alpha} y_b\right) C_{n-1}^{\eta_s+1}(\xi_b) \right] C_n^{\eta_s}(\xi_a) \chi_{-s} \chi_s^\dagger \right\} \\
- \left\{ \left[\left(-\omega_{n,s}^{(dS)} + m\right) C_{n-s}^{\eta_s+s}(\xi_b) C_{n-s}^{\eta_s+s}(\xi_a) \chi_{-s} \chi_{-s}^\dagger \right.\right. \\
+ \sqrt{\alpha} \left[-\left(\frac{1+s}{2}\right)\left(1 + \frac{m\omega}{\alpha}\right) \tan\left(\sqrt{\alpha} y_b\right) C_{n-s}^{\eta_s+s}(\xi_b) \right. \\
\left.\left. +2\left(\frac{1+s}{2} + \frac{m\omega}{\alpha}\right) \cos\left(\sqrt{\alpha} y_b\right) C_{n-s-1}^{\eta_s+s+1}(\xi_b) \right] C_{n-s}^{\eta_s+s}(\xi_a) \chi_s \chi_{-s}^\dagger \right\}.
\end{aligned} \tag{56}
$$

From above expression, we can rewrite the causal Green's function as follows:

$$
\begin{aligned}
S^{(AdS)}(x_b, x_a) = -\imath \sum_{s=\pm 1} \sum_n \exp\left(-\imath s \omega_{n,s}^{(AdS)}(y_{0b}-y_{0a})\right) \Theta\left(s\left(y_{0b}-y_{0a}\right)\right) \\
\times \left[\Gamma(\eta_s) \sqrt{\frac{2^{2\eta_s-1} n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}} \sqrt{\frac{m+\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_n^{\eta_s}(u_b) \vartheta_b^{\eta_s} \chi_s \right. \\
\left. +\imath\sqrt{\alpha}\Gamma(\eta_s+s) \sqrt{\frac{2^{2(\eta_s+s)-1}(n-s)!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s+s)}} \sqrt{\frac{m-\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_{n-s}^{\eta_s+s}(u_b) \vartheta_b^{\eta_s+s} \chi_{-s} \right] \\
\times \left[\Gamma(\eta_s) \sqrt{\frac{2^{2\eta_s-1} n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}} \sqrt{\frac{m+\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_n^{\eta_s}(u_a) \vartheta_a^{\eta_s} \chi_s^\dagger \right. \\
\left. +\imath\sqrt{\alpha}\Gamma(\eta_s+s) \sqrt{\frac{2^{2(\eta_s+s)-1}(n-s)!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s+s)}} \sqrt{\frac{m-\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_{n-s}^{\eta_s+s}(u_a) \vartheta_a^{\eta_s+s} \chi_{-s}^\dagger \right].
\end{aligned} \tag{57}
$$

In Eq. (57) we have two types of propagation, one with positive energy ($+E_{n,\alpha}^{Anti}$) propagating to the future and the other with negative energy ($-E_{n,\alpha}^{Anti}$) propagating to the past. Consequently, we obtain this result in the former variable,

$$
S^{(\alpha)}(x_b, x_a, t_b - t_a) = -\sum_{s=\pm 1} \sum_n \begin{bmatrix} \Theta(t_b - t_a) \Psi_n^{(\alpha)+}(x_b) \bar{\Psi}_n^{(\alpha)+}(x_a) e^{-\imath E_{n,\alpha,s}^{Anti}(t_b-t_a)} + \\ \Theta(-(t_b - t_a)) \Psi_n^{(\alpha)-}(x_b) \bar{\Psi}_n^{(\alpha)-}(x_a) e^{\imath E_{n,\alpha,s}^{Anti}(t_b-t_a)} \end{bmatrix}. \tag{58}
$$

This formula is the spectral decomposition of the Green function, within which we extract the wave functions

$$
\begin{aligned}
\Psi_n^{(AdS)s}(x) = \Gamma(\eta_s) \sqrt{\frac{2^{2\eta_s-1} n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}} \sqrt{\frac{m+\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_n^{\eta_s}(u_b) \vartheta_b^{\eta_s} \chi_s \\
+ \imath\sqrt{\alpha}\Gamma(\eta_s+s) \sqrt{\frac{2^{2(\eta_s+s)-1}(n-s)!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s+s)}} \sqrt{\frac{m-\omega_{n,s}^{(AdS)}}{2\omega_{n,s}^{(AdS)}}} C_{n-s}^{\eta_s+s}(u_b) \vartheta_b^{\eta_s+s} \chi_{-s},
\end{aligned} \tag{59}
$$

and we can return to the old variables by means of the following relations

$$
u = \sin\left(\arctan\left(\sqrt{\alpha} x\right)\right), \ \vartheta = \cos\left(\arctan\left(\sqrt{\alpha} x\right)\right). \tag{60}
$$

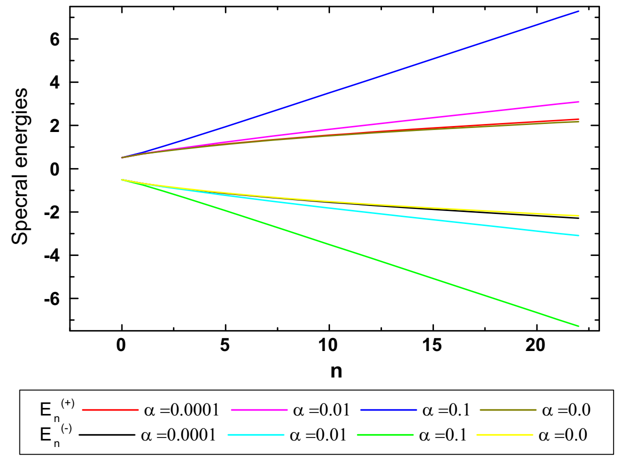

Fig. 1 $E_{n,s=1}^{(AdS)}$ is the energy spectrum versus n for several values of α

Where the corresponding spectral energies are

$$E_{n,\alpha,s}^{(AdS)\pm} = \pm\sqrt{m^2 + 2m\omega\left(n + \frac{1-s}{2}\right) + \alpha\left(n^2 + \frac{1-s}{2}(2n+1)\right)}. \tag{61}$$

The dependence on n^2 corresponding to α effect of the modification of the Heisenberg algebra, due to the EUP, which is a characteristic of the confinement phenomena. With various the values of α and with spin up ($s = 1$), we can plot the appropriate curves of positive and negative energies in Fig. 1. We clearly notice that the energy $E_{n,\alpha,s}^{(AdS)\pm}$ is presented as a function of n for several values of α, the spectrum is expanded, $E_{n,\alpha,s}^{(AdS)+}$ is an increasing ($E_{n,\alpha,s}^{(AdS)-}$ is decreasing) monotonous function for arbitrary α.

Next, we want to check the current density (ρ, J_x) for $(1+1)$-dimensional Dirac oscillator in the context of the EUP. Activating the positive use of this method (path integral formalism) for normalized the wave functions in the context of the extended uncertainty principle. As we know the current density are defined as

$$\rho = \int d_\lambda x (\Psi_n^{(AdS)s}(x))^\dagger \Psi_n^{((AdS))s}(x), \tag{62}$$

$$J_x = \int d_\lambda x \bar{\Psi}_n^{((AdS))s}(x)\gamma^1\Psi_n^{((AdS))s}(x). \tag{63}$$

After straightforward calculation, we can confirm the current density of Dirac oscillator in $(1+1)$ dimension in the context of the EUP are given as

$$\rho = 1, \tag{64}$$

$$J_x = \int d_\lambda x (\Psi_n^{(AdS)s}(x))^\dagger \sigma^2 \Psi_n^{(AdS)s}(x) = 0. \tag{65}$$

Which approves the same results in usual case of the Dirac oscillator in $(1+1)$ dimension $(\alpha = 0)$.

6 de Sitter Snyder Spaces

In the case of de-Sitter Snyder space, we will follow the same calculation procedures as presented in the previous section. Which can be constructed by replacing α by $(-\alpha')$ in Eq. (61). The spectral energies $E_{n,\alpha',s}^{(dS)\pm}$ are given as:

$$E_{n,\alpha',s}^{(dS)\pm} = \pm\sqrt{m^2 + 2m\omega\left(n + \frac{1-s}{2}\right) - \alpha'\left(n^2 + \frac{1-s}{2}(2n+1)\right)}. \tag{66}$$

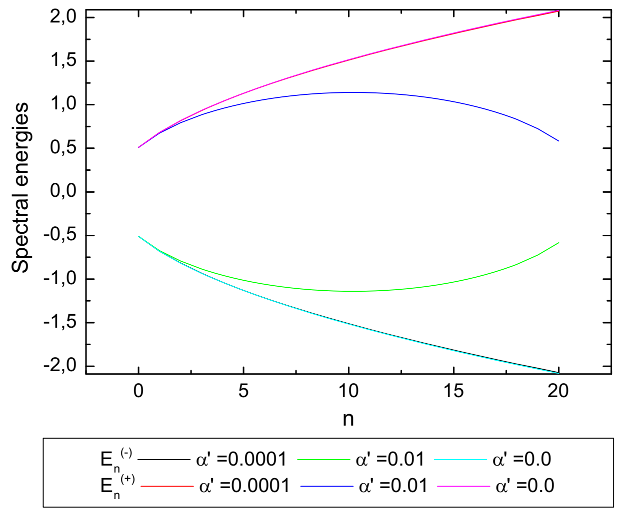

Fig. 2 $E^{(dS)}_{n,s=1}$ is the energy spectrum versus n for several values of α'

We note that when the quantum number n is large, the spectral energies would have no physical meaning. This indicates that one needs to impose an upper bound on the values of n. From these last expressions of the spectral energies $E^{(dS)\pm}_{n,\alpha',s}$, we can determine this limit by using

$$\frac{E^{(dS)}_{n,\alpha',s}}{dn}\,|_N = 0, \tag{67}$$

where N implies to $\left(\frac{m\omega}{\alpha'}+\frac{1-s}{2\alpha'}\right)$, and $E^{(dS)+}_{n,\alpha',s}$ is decreasing (while $E^{(dS)-}_{n,\alpha',s}$ is an increasing) monotonous function for arbitrary α'. These cases are illustrated by the following curve (Fig. 2)

While the corresponding wave functions are given from Eq. (59) by substituting ($\alpha \to -\alpha'$), and which leads to

$$u = \sinh\left(\tanh^{-1}\left(\sqrt{\alpha'}x\right)\right),\ \vartheta = \cosh\left(\tanh^{-1}\left(\sqrt{\alpha'}x\right)\right). \tag{68}$$

In the following subsections, we will present the special and important cases to validate these our calculations.

6.1 Without Deformation Case

In order to obtain the ordinary case, we put the limit $\alpha \to 0$, $\eta_s = \eta \to \infty$, by using [46]

$$\lim_{\lambda\to\infty} \lambda^{-\frac{n}{2}} C_n^{\frac{\lambda}{2}}\left(x\sqrt{\frac{2}{\lambda}}\right) = \frac{2^{-\frac{n}{2}}}{n!}H_n(x),\quad \lim_{\lambda\to\infty}\frac{\Gamma(\lambda+a)}{\Gamma(\lambda)}e^{-a\ln\lambda} = 1, \tag{69}$$

the doubling formula

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\Gamma\left(x+\frac{1}{2}\right), \tag{70}$$

and

$$\lim_{\alpha\to 0,\eta\to\infty}\left(1+\alpha x^2\right)^{\eta} = \exp\left(-\frac{m\omega}{2}x^2\right). \tag{71}$$

From the above limits, we can obtain the wave functions and energy spectrum, and they are given respectively

$$\lim_{\alpha\longrightarrow 0}\Psi_n^{(\alpha)s}(x) = \Psi_n^{(\alpha=0)s}(x) = \begin{pmatrix} f_n^{(\alpha=0)s}(x) \\ g_n^{(\alpha=0)s}(x)\end{pmatrix},$$

with the following components,

$$f_n^{(\alpha=0)s}(x)=\sqrt{\frac{\sqrt{m\omega/\pi}\left(E_{n,s}^{(\alpha=0)}+m\right)}{2^{n+1}n!E_{n,s}^{(\alpha=0)}}}\exp\left(-\frac{m\omega}{2}x^2\right)H_n\left(\sqrt{m\omega}x\right). \tag{72}$$

$$g_n^{(\alpha=0)s}(x)=-\sqrt{\frac{\sqrt{m\omega/\pi}\left(E_{n,s}^{(\alpha=0)}-m\right)}{2^{n}(n-s)!E_{n,s}^{(\alpha=0)}}}\exp\left(-\frac{m\omega}{2}x^2\right)H_{n-s}\left(\sqrt{m\omega}x\right), \tag{73}$$

and

$$E_{n,\alpha'=0,s}^{(dS)\pm}=\pm\sqrt{m^2+2m\omega\left(n+\frac{1-s}{2}\right)}. \tag{74}$$

We deduct exactly the same result without deformed uncertainty relation which coincide with those obtained from the usual Dirac oscillator in $(1+1)$ dimensions [11].

6.2 Non-relativistic Limit

To obtain the energy level in non relativistic limit case for the one-dimensional Dirac oscillator in anti-de Sitter spaces system $E_{NR,s}^{(AdS)}$, we put $E_{n,\alpha,s}^{(AdS)}=m+E_{NR,s}^{(AdS)}$ with $m\gg E_{NR,s}^{(AdS)}$ and using the Taylor development of (61) in the second order approximation, we find:

$$\begin{aligned}E_{NR,s}^{(AdS)+}\approx{}&\omega\left(n+\frac{1-s}{2}\right)+\frac{\alpha}{2m}\left(n^2+\frac{1-s}{2}(2n+1)\right)\\&+\frac{1}{2}\left[\omega\left(n+\frac{1-s}{2}\right)+\frac{\alpha}{2m}\left(n^2+\frac{1-s}{2}(2n+1)\right)\right]^2,\end{aligned} \tag{75}$$

with m represents the rest energy of the particle, the second and third terms represent, respectively, the energy of the non-relativistic oscillator of frequency ω and the relativistic correction both in the context of the extended uncertainty principle.

This implies that the corresponding eigenvalues associated with this energy level in the non-relativistic limit are given by

$$\Psi_{NR,s}^{(AdS)+}(x)=\Gamma(\eta_s)\sqrt{\frac{2^{2\eta_s-1}n!(n+\eta_s)\sqrt{\alpha}}{\pi\Gamma(n+2\eta_s)}}C_n^{\eta_s}(u_b)\,\vartheta_b^{\eta_s}\chi_s, \tag{76}$$

where we have used the following limits:

$$\lim_{m\gg}\sqrt{\frac{m+\omega_{n,s}^{(AdS)+}}{2\omega_{n,s}^{(AdS)+}}}\approx 1,\ \lim_{m\gg}\sqrt{\frac{m-\omega_{n,s}^{(AdS)+}}{2\omega_{n,s}^{(AdS)+}}}\approx 0. \tag{77}$$

7 Conclusion

In this paper, we have constructed the path integral representation for the Green function for the Dirac oscillator in $(1+1)$ dimension in the EUP. Which indicates to presence a nonzero minimal uncertainty in momentum. We obtained the exact spectral energies and corresponding eigenfunctions expressed in terms of Gegenbauer polynomials. The energy levels show a dependence on n^2 corresponding to α effect of the modification of the Heisenberg algebra, due to the EUP, which is a characteristic of the confinement phenomena as in the case of non-commutative geometry. As a result, for a fixed value of n, the energy $E_{n,\alpha,+}^{(AdS)}$ increases monotonically with the increase of the EUP parameter α.

We have also deduced special cases:

(1) de-Sitter spaces case by replacing $\alpha \longrightarrow -\alpha'$, we note that the corresponding energy spectrum $E^{(dS)}_{n,\alpha',s}$ would have an unphysical behavior when the quantum number n is large. This indicates that one needs to impose an upper bound on the values of n and we can also see that the energy spectrum on the de Sitter space is smaller than the energy in ordinary quantum mechanics.

(2) Absence of deformation case by taking the limit ($\alpha \longrightarrow 0$), we obtain the usual Heisenberg algebra. The same result without deformed uncertainty relation which has been done by Rekioua and Boujdedaa [11]. A generalization of this work in the presence of an electromagnetic field that requires a thorough discussion is currently under consideration, and will be the subject of another study. At the end of this paper, it is worth mentioning that the results obtained make it possible to detect the effects due to the large scale curvature of spacetime on some physical systems: for example the confinement of quarks in quantum chromodynamics (QCD) and the description of certain properties of electrons in graphene. We recall that the dynamics of these two physical examples cited is modeled by the relativistic Dirac oscillator, as it is known in the literature [36,47].

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Relativistic oscillators in new type of the extended uncertainty principle

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We present the exact solutions of one-dimensional Klein–Gordon and Dirac oscillators subject to the uniform electric field in the context of the new type of the extended uncertainty principle using the displacement operator method. The energy eigenvalues and eigenfunctions are determined for both cases. For the Klein–Gordon oscillator case, the wave functions are expressed in terms of the associated Laguerre polynomials and for the Dirac oscillator case, the wave functions are obtained in terms of the confluent Heun functions. The limiting cases are also studied using the special values of the physical parameters.

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1. Introduction

The dynamics of some physical systems are modeled by deformed algebras. For example, the description of the low energy excitations of graphene and the Fermi velocity are based on a deformation of the Heisenberg algebra which makes the commutator of momenta proportional to the pseudo-spin.[1] The dynamics of systems with variable masses in semiconductor heterostructures are formulated by deformed quadratic algebra[2] and a deformed Heisenberg algebra for the motion of a ^{3}He impurity atom in the Bose liquid is suggested in Ref. 3. In the context of quantum gravity, the usual Heisenberg uncertainty principle can be replaced by the so-called generalized uncertainty principle (GUP)[4–10] and it is characterized by the existence of a minimal length scale in the order of the Planck length. Several research fields in which the concept of minimal length plays an essential role are, the string theory,[11] noncommutative geometries,[12] black hole physics[13] and quantum gravity.[14] Recently, in this sense, this GUP has undergone notable development based on some physical observations and a significant number of papers have been published in diverse physics area, citing for instance: the modification of the black hole thermodynamics,[15,16] the corrections to the Unruh effect and related Unruh temperature,[17] beyond the linear dispersion relations of graphene,[18] the energy-dependent potentials,[19] correction of nonthermal radiation spectrum in the background of noncommutative geometry,[20] an explicit construction for Gazeau–Klauder coherent states for a non-Hermitian system on a noncommutative,[21] the generalized time-dependent q-deformed coherent states for a noncommutative harmonic oscillator,[22] entangled states in a noncommutative space with the squeezed states,[23] q-deformed nonlinear coherent states and nonclassical behaviors of q-deformed version of the Schrodinger cat states in noncommutative space[24] and an experimental realization of effects of noncommutative theories.[25] In addition, if we consider the quantum effects due to the topology of the physical space, the specific type of the modified uncertainty principle is called the extended uncertainty principle (EUP)[26–31] and it is characterized by the existence of a nonzero minimal uncertainty in momentum. As an example, Mignemi showed that in a (anti-)de Sitter background, the Heisenberg uncertainty principle is modified by introducing corrections proportional to the cosmological constant $\Lambda = -3\lambda^2$, where $\lambda^2 < 0$ for de Sitter space–time, and $\lambda^2 > 0$ for the anti-de Sitter space–time. The introduction of this idea of (EUP) has drawn great interest and many papers have appeared in the literature to address the effects of the extended commutation relations, the thermodynamic properties of the Schwarzschild black hole and Unruh effect by using the simplest form of the EUP are investigated in Ref. 32, the corrections to Hawking temperature and Bekenstein entropy of a black hole for Rindler and cosmological horizons,[33] the analytical solution of the pseudoharmonic potential for N_2 and CO diatomic molecules is determined and it is claimed that the energy corrections coming from the deformation parameter are unlikely to be detectable experimentally,[30] the thermodynamic properties of the relativistic harmonic oscillators are investigated[34] signals of the weak and strong deflection gravitational lensings are studied,[35] the quantum gravity effects in the vicinity of a black hole,[36] the Ramsauer–Townsend effect in q-deformed quantum mechanics[37] and the Klein–Gordon oscillator in an uniform magnetic field.[38]

Furthermore, in the past few years, another new type of EUP with a minimum momentum dispersion has been introduced by the action of the translation operator in a space with a diagonal metric for the purpose of describing the motion of a quantum particle in curved space.[39–43,46–51] One has

$$T_\lambda(\delta x)|x\rangle = |x + \delta x + \lambda x\delta x\rangle\,, \tag{1}$$

where δx is an infinitesimal displacement and the parameter λ is the inverse of a characteristic length that determines the mixing between the displacement and the original position state. This translation is nonadditive and it can be written up to the first order in δx as

$$T_\lambda(\delta x) = 1 - \frac{i\delta x}{\hbar}P_\lambda\,, \tag{2}$$

where P_λ is a generalized momentum operator. This property changes the commutation relation for position and momentum as

$$[\hat{x}, P_\lambda] = i\hbar(1 + \lambda x)\,, \tag{3}$$

and yields a generalized uncertainty relation

$$\Delta x\Delta P_\lambda \geq \frac{\hbar}{2}(1 + \lambda\langle x\rangle)\,. \tag{4}$$

The generalized momentum operator and the position operators satisfying Eq. (3) can be represented by[40–42]

$$P_\lambda = -i\hbar(1 + \lambda x)\frac{d}{dx} \quad \text{and} \quad \hat{x} = x\,, \tag{5}$$

and in Hermitian form by[43]

$$P_\lambda = -i\hbar D_\gamma \quad \text{and} \quad \hat{x} = x\,. \tag{6}$$

Here,

$$D_\gamma = \left[(1 + \lambda x)\frac{d}{dx} + \frac{\lambda}{2}\right]. \tag{7}$$

On the other hand, the nonadditive operator corresponds to the infinitesimal generator of the q-exponential function[44,45]

$$\exp_q(x) \equiv [1 + (1 - q)x]^{\frac{1}{1-q}}\,, \tag{8}$$

where x is a dimensionless variable and $\lambda \equiv (1 - q)$. Equation (8) represents a fundamental mathematical definition for the generalized thermostatistics of Tsallis and its applications.[46–51] For this purpose to see what kind of physical importance the translation operator bears within this framework, some problems were solved for a quantum system. For example, the study of a particle under a null potential confined in a square well,[40–43] the solution of the quantum harmonic oscillator where the problem is converted to the Morse potential case,[41,52] the position-dependent mass system with a variable potential[53] and, Arda used this displacement operator to study the particle moving in an inverse square plus Coulomb-like potential which is similar to the Rosen–Morse potential in usual position space,[54] a deformed Bohmian formalism by means of a deformed Fisher information functional and a derivation a deformed Cramer–Rao bound in Ref. 55, a displaced anisotropic two-dimensional non-Hermitian harmonic oscillator and graphics for the specific heat and for the entropy of both oscillators compared with several experiments in Ref. 56, the classical mechanics in the curved space and Bohr–Sommerfeld quantization[39] and a particle confined in a bidimensional box within a generalized space.[57]

The main purpose of this paper is to study the Klein–Gordon and Dirac oscillators with a uniform electric field analytically in the context of this new type of EUP using the displacement operator method. To the best of our knowledge, no relativistic problem has been studied within this framework of the translation operator. Consequently, our attempt is to approach this new type of EUP for a relativistic problem and to study the influence of this deformation on the properties of the systems, such as the confinement phenomenon and energy value of the Stark shift.

The rest of the paper is organized as follows. In Sec. 2, we give the exact solution of the Klein–Gordon oscillator equation with a uniform electric field. The case of the Dirac oscillator with a uniform electric field is treated in Sec. 3. Some limiting cases of both solutions are also studied using the special values of the physical parameters.

2. Klein–Gordon Oscillator Equation with a Uniform Electric Field

In regular space, the Klein–Gordon oscillator subject to an electric field Θ_{KG} in one-dimensional space is defined by

$$\Theta_{\mathrm{KG}}\psi(x) = \left[(\hat{p}+im\omega\hat{x})(\hat{p}-im\omega\hat{x}) + m^2 - (E-q\varepsilon\hat{x})^2\right]\psi(x) = 0\,, \tag{9}$$

which can be written as

$$\left\{p^2 + \left(m^2\omega^2-\varepsilon^2\right)x^2 + im\omega[x,p] + 2\varepsilon Ex - \left(E^2-m^2\right)\right\}\psi(x) = 0\,, \tag{10}$$

where q is the electrical charge and ε is the intensity of electric field. Note that we use the units where $\hbar = c = 1$.

The continuity equation can be deduced from the modified Klein–Gordon equation (9) and its conjugate by the relation,

$$\frac{\partial\rho}{\partial t} + D_\gamma J_\gamma = 0\,, \tag{11}$$

with

$$\rho = i\left(\Psi^*\partial_t\Psi - \Psi\partial_t\Psi^*\right), \tag{12}$$

and J_γ is the modified current density:

$$J_\gamma = -i\left(\Psi^*(1+\gamma x)\frac{d\Psi}{dx} - \Psi(1+\gamma x)\frac{d\Psi^*}{dx}\right). \tag{13}$$

In order to solve Eq. (10)), we use the transformation,

$$u = (1+\gamma x)\,, \tag{14}$$

and using the representations (6) and (3), Eq. (10) becomes

$$\begin{aligned}\Bigg\{\frac{d^2}{du^2} + \frac{2}{u}\frac{d}{du} + \left(\frac{1}{4} - \frac{(m^2\omega^2-\varepsilon^2)}{\gamma^4} + \frac{2\varepsilon E}{\gamma^3} + \frac{(E^2-m^2)}{\gamma^2}\right)\frac{1}{u^2} \\ + \left(\frac{2(m^2\omega^2-\varepsilon^2)}{\gamma^4} + \frac{m\omega}{\gamma^2} - \frac{2\varepsilon E}{\gamma^3}\right)\frac{1}{u} + \frac{(\varepsilon^2-m^2\omega^2)}{\gamma^4}\Bigg\}\psi(u) = 0\,. \end{aligned}\tag{15}$$

Introducing the notations

$$\begin{aligned}\delta &= \frac{1}{4} - \frac{(m^2\omega^2-\varepsilon^2)}{\gamma^4} + \frac{2\varepsilon E}{\gamma^3} + \frac{(E^2-m^2)}{\gamma^2}\,,\\ \eta &= \left(\frac{2(m^2\omega^2-\varepsilon^2)}{\gamma^4} + \frac{m\omega}{\gamma^2} - \frac{2\varepsilon E}{\gamma^3}\right),\\ \zeta &= \frac{\sqrt{(m^2\omega^2-\varepsilon^2)}}{\gamma^2} \quad \text{with} \quad m\omega > \varepsilon\,,\end{aligned}\tag{16}$$

we get

$$\psi'' + \frac{2}{u}\psi' + \left(\frac{\delta}{u^2} + \frac{\eta}{u} - \zeta^2\right)\psi = 0\,. \tag{17}$$

To simplify Eq. (17), we introduce

$$\psi(u) = u^\sigma \exp(-\zeta u)F(u)\,, \quad u \mapsto y = 2\zeta u\,, \tag{18}$$

where σ is a constant to be determined later. After using (18), the differential equation (17) will reduce to the equation of the associated Laguerre polynomials $L_n^k(y)$,

$$\left[y\frac{d^2}{dy^2} + [(2\sigma+2)-y]\frac{d}{dy} + \frac{1}{y}[\sigma(\sigma-1)+2\sigma+\delta] + \frac{1}{2\zeta}[\eta - 2\zeta - 2\zeta\sigma]\right]F(y) = 0\,. \tag{19}$$

We impose the constraint,

$$\sigma(\sigma-1) + 2\sigma + \delta = 0\,, \tag{20}$$

to eliminate the coefficient proportional to $\frac{1}{y}$, and

$$\begin{cases}\frac{1}{2\zeta}[\eta - 2\zeta - 2\zeta\sigma] = n\,,\\ 2\sigma + 2 = k + 1\,.\end{cases}\tag{21}$$

The relation (20) leads to the following expressions for σ:

$$\sigma_{\pm} = -\frac{1}{2} \pm \frac{1}{\gamma}\sqrt{m^2 - E^2 + \frac{m^2\omega^2 - \varepsilon^2}{\gamma^2} - \frac{2E\varepsilon}{\gamma}}\,. \tag{22}$$

Among these two solutions, the physically acceptable one is only σ_+. To extract the energy spectrum, we substitute the expression (22) into the first relation of (21). Then it is straightforward to show that

$$E^{\pm} = -\frac{\varepsilon\gamma}{2m\omega}[(2n+1)\Omega - 1] \pm \Omega\sqrt{m^2 + m\omega[(2n+1)\Omega - 1] - \frac{\gamma^2}{4}[(2n+1)\Omega - 1]^2}\,, \tag{23}$$

with $\Omega = \frac{\sqrt{(m^2\omega^2 - \varepsilon^2)}}{m\omega}$. We should note that the expression of energy spectrum contains all corrections of all orders of $(\varepsilon\gamma)^2$. This is related to the exact contribution to the Stark effect in this framework of the deformation. On the other hand, it varies by the power of n^2, which explains the confinement phenomenon. For large values of n, the square of the energy spectrum $(E)^2$ becomes negative. Thus, in order to ensure the positivity of the square of the energy, one must impose an upper bound on the allowed values of n.

Expanding up to the first order in γ^2, we obtain

$$\begin{aligned} E^{\pm} &= \pm\Omega\sqrt{m^2 + m\omega[(2n+1)\Omega - 1]} - \frac{\varepsilon\gamma}{2m\omega}[(2n+1)\Omega - 1] \\ &\quad \mp \frac{\gamma^2\Omega[(2n+1)\Omega - 1]^2}{8\sqrt{m^2 + m\omega[(2n+1)\Omega - 1]}}\,. \end{aligned} \tag{24}$$

The first term in (24) is the energy spectrum of the usual Klein–Gordon oscillator subject to the uniform electric field. The second and the third terms represent the quantum fluctuations due to the new type of EUP. It is remarkable that the expression of the energy spectrum contains additional deformed correction terms depending on the deformation parameter γ, γ^2 and with powers in n^2, which explains the phenomenon of confinement. We can see that the energy spectrum in the context of this deformation is smaller than the energy in the ordinary case.

Solving Eq. (9) along with the relations (18), (19) and (22), we obtain the final form of the wave function in the former variable x as

$$\begin{aligned} \psi(x) &= N_{nr}(1+\gamma x)^{-\frac{1}{2}+\frac{1}{\gamma}\sqrt{m^2 - E^2 + \frac{m^2\omega^2-\varepsilon^2}{\gamma^2} - \frac{2E\varepsilon}{\gamma}}} \\ &\quad \times \exp\left\{-\frac{1}{\gamma^2}\sqrt{(m^2\omega^2 - \varepsilon^2)}(1+\gamma x)\right\} \\ &\quad \times L_n^{\frac{2}{\gamma}\sqrt{m^2 - E^2 + \frac{m^2\omega^2-\varepsilon^2}{\gamma^2} - \frac{2E\varepsilon}{\gamma}}}\left(\frac{2}{\gamma^2}\sqrt{(m^2\omega^2 - \varepsilon^2)}(1+\gamma x)\right), \end{aligned} \tag{25}$$

and N_{nr} is a normalization constant.

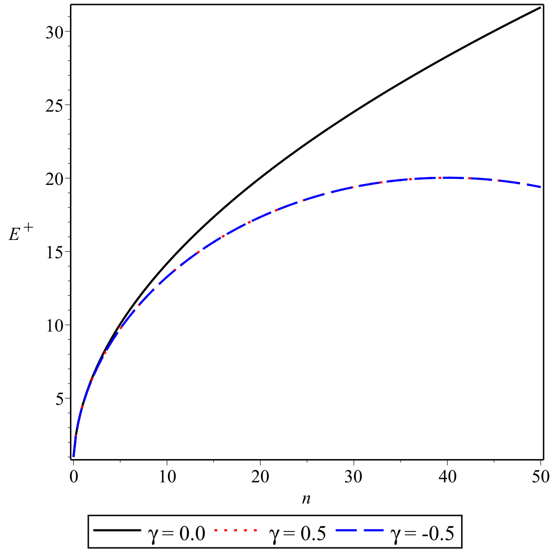

Fig. 1. (E^+ versus n) for $\varepsilon = 0$ (Klein–Gordon oscillator).

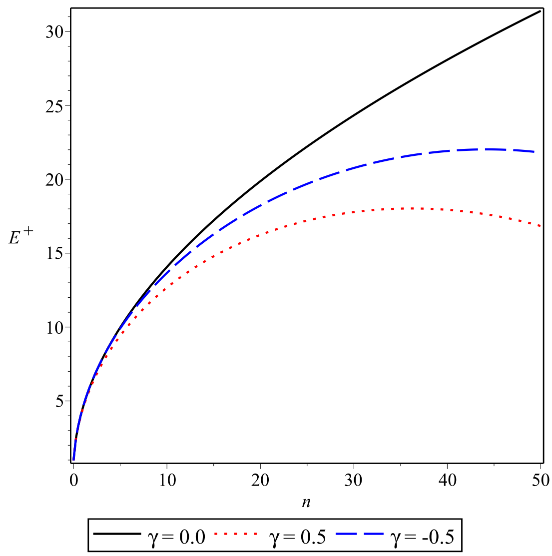

Fig. 2. (E^+ versus n) for $\varepsilon = 1$ (Klein–Gordon oscillator).

We can present our results graphically for some numerical values of the physical parameters. We will take $m = 1$ and $\omega = 10$ in our analysis. We will plot the curves only for E^+ as the curves for E^- do not show a different physical behavior.

In Fig. 1, we plot the energy levels as a function of quantum number n for various values of γ and for $\varepsilon = 0$. We see that the values for nonzero γ coincide. If we take a fixed but nonzero ε as in Fig. 2, we find that the energy behavior is

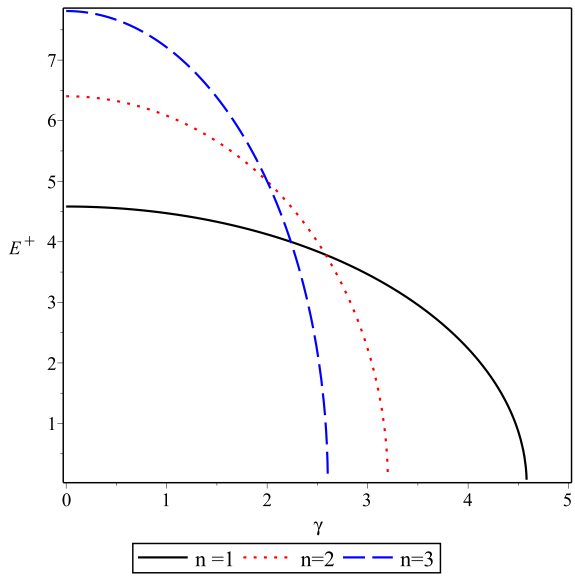

Fig. 3. (E^+ versus γ) for $\varepsilon = 0$ (Klein–Gordon oscillator).

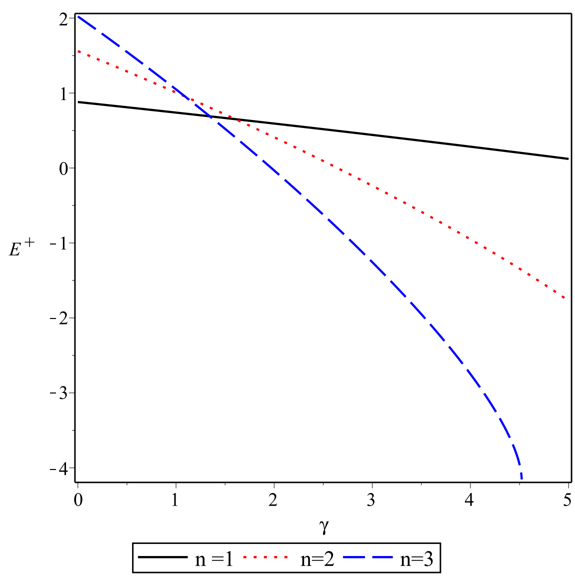

Fig. 4. (E^+ versus γ) for $\varepsilon = 9$ (Klein–Gordon oscillator).

different. The nonzero electric field yields a physical effect on the system. Figures 3 and 4 show the behavior of the energy for varying γ and for a fixed ε (we used $\varepsilon = 0$ and $\varepsilon = 9$, respectively). Here, we see the effect of γ on the energy behavior for some fixed n values.

2.1. *Some special cases*

We can consider some special cases for vanishing γ and ε.

For $\gamma = 0$, namely, in the absence of deformation, we replace $\gamma = 0$ in (23):

$$E^{\pm} = \mp\Omega\sqrt{m^2 - m\omega + (2n+1)m\omega\Omega}\,. \tag{26}$$

The case for $\varepsilon = 0$, namely, in the absence of an electric field, implies $\Omega = 1$, and the expression of the energy spectrum (23) becomes

$$E^{\pm} = \pm\sqrt{-\gamma^2 n^2 + m^2 + 2nm\omega}\,. \tag{27}$$

In the case where $\gamma = \varepsilon = 0$, we have the pure Klein–Gordon oscillator case. This limit yields

$$E^{\pm} = \pm\sqrt{m^2 + 2nm\omega}\,, \tag{28}$$

which is in agreement with the result of the ordinary case.

3. Dirac Oscillator Equation with a Uniform Electric Field

The Dirac oscillator with a uniform electric field is defined by the expression,[58,59]

$$[\alpha(\hat{p} - im\omega\beta\hat{x}) + \beta m]\Psi(x) = (E - q\varepsilon\hat{x})\Psi(x)\,, \tag{29}$$

where $\Psi(x) = \binom{\phi_1(x)}{\phi_2(x)}$ and α, β are the Dirac matrices given by

$$\alpha = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{30}$$

Note that we are using the units where ($\hbar = c = 1$). Using the matrices (30) and the definition of $\Psi(x)$ in Eq. (29), we obtain the system,

$$\begin{cases} (p_x + im\omega x)\phi_2 = (E - m - \varepsilon x)\phi_1(x)\,, \\ (p_x - im\omega x)\phi_1 = (E + m - \varepsilon x)\phi_2(x)\,. \end{cases} \tag{31}$$

Introducing the notation $\Pi^{\pm} = p_x \pm im\omega x$ and $M^{\pm} = E \pm m - \varepsilon x$, the new form of the system (31) can be obtained as

$$\begin{cases} \Pi^{+}\phi_2(x) = M^{-}\phi_1(x)\,, \\ \Pi^{-}\phi_1(x) = M^{+}\phi_2(x)\,. \end{cases} \tag{32}$$

In order to decouple the above system, we write ϕ_2 in terms of ϕ_1,

$$\phi_2(x) = \left(M^{+}\right)^{-1}\Pi^{-}\phi_1(x)\,, \tag{33}$$

and we replace it in the first equation as

$$\Pi^{+}\left(M^{+}\right)^{-1}\Pi^{-}\phi_1(x) = M^{-}\phi_1(x)\,,$$

using

$$\Pi^{+}\left(M^{+}\right)^{-1} = \left(M^{+}\right)^{-1}\Pi^{+} + \left[\Pi^{+}, \left(M^{+}\right)^{-1}\right]. \tag{34}$$

Then we multiply the whole equation by M^{+} on the left to get

$$\left[\Pi^{+}\Pi^{-} - M^{+}M^{-} + M^{+}\left[\Pi^{+}, \left(M^{+}\right)^{-1}\right]\Pi^{-}\right]\phi_1(x) = 0\,, \tag{35}$$

where $[\cdot\,,\cdot]$ is the commutator between two operators.

We note that the first two terms represent exactly the Klein–Gordon oscillator. We use

$$\Theta_{\mathrm{KG}} = \Pi^{+}\Pi^{-} - M^{+}M^{-}\,, \tag{36}$$

where

$$\Theta_{\mathrm{KG}} = (\hat{p} + im\omega\hat{x})(\hat{p} - im\omega\hat{x}) + m^2 - (E - q\varepsilon\hat{x})^2\,, \tag{37}$$

and the third term characterizes the spinor effect of the particle. Using the definitions, Eq. (35) can be written as

$$\left\{\Theta_{\mathrm{KG}} + (E + m - \varepsilon x)\left[(p_x + im\omega x), \frac{1}{(E + m - \varepsilon x)}\right](p_x - im\omega x)\right\}\phi_1(x) = 0\,. \tag{38}$$

By a direct calculation, Eq. (38) becomes

$$\left\{\Theta_{\mathrm{KG}} - \frac{i\varepsilon(1 + \gamma x)}{(E + m - \varepsilon x)}(p_x - im\omega x)\right\}\phi_1(x) = 0\,, \tag{39}$$

where we used Eq. (3).

To solve Eq. (39), we use the change of variable (14). Then we obtain

$$\left\{\frac{d^2}{du^2} + \left(\frac{2}{u} + \frac{1}{r - u}\right)\frac{d}{du} + \frac{\eta}{u} + \frac{m\omega}{\gamma^2}\frac{1}{(r - u)} + \frac{\delta}{u^2} + \frac{\tau}{u(r - u)} - \zeta^2\right\}\phi_1(u) = 0\,, \tag{40}$$

where

$$\begin{aligned}
\delta &= \frac{1}{4} - \frac{(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{2\varepsilon E}{\gamma^3} + \frac{(E^2 - m^2)}{\gamma^2}\,,\\
\eta &= \left(\frac{2(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{m\omega}{\gamma^2} - \frac{2\varepsilon E}{\gamma^3}\right),\\
\tau &= \left(\frac{1}{2} - \frac{m\omega}{\gamma^2}\right),\\
\zeta^2 &= \frac{\sqrt{(m^2\omega^2 - \varepsilon^2)}}{\gamma^2} \quad \text{with} \quad m\omega > \varepsilon\,,\\
r &= \frac{\gamma(E + m)}{\varepsilon} + 1\,.
\end{aligned} \tag{41}$$

In order to simplify Eq. (40), we introduce

$$\phi_1(u) = u^\sigma \exp(-\zeta u) G(u)\,, \tag{42}$$

where σ is a constant to be determined letter. We obtain

$$\left\{\frac{d^2}{du^2} + \left(\frac{2\sigma+2}{u} + \frac{1}{r-u} - 2\zeta\right)\frac{d}{du} + \frac{1}{u^2}(\sigma(\sigma-1)+2\sigma+\delta) + \frac{1}{u}(-2\zeta\sigma+\eta-2\zeta) + \frac{1}{(r-u)}\left(\frac{m\omega}{\gamma^2}-\zeta\right) + \frac{1}{u(r-u)}(\sigma+\tau)\right\}G(u) = 0\,. \tag{43}$$

To reduce this equation to a class of known differential equation with a polynomial solution, we need to eliminate the coefficient proportional to $\frac{1}{u^2}$. We impose

$$\sigma(\sigma-1) + 2\sigma + \delta = 0\,, \tag{44}$$

and this leads to the expression

$$\sigma_\pm = -\frac{1}{2} \pm \frac{1}{\gamma}\sqrt{m^2 - E^2 + \frac{m^2\omega^2 - \varepsilon^2}{\gamma^2} - \frac{2E\varepsilon}{\gamma}}\,. \tag{45}$$

Among these two solutions, the physically acceptable one is only σ_+, and the second solution leads to a nonphysical wave function. We introduce $z = \frac{u}{r}$, then Eq. (43) takes the form

$$\left\{\frac{d^2}{dz^2} + \left(\frac{2\sigma+2}{z} - \frac{1}{z-1} - 2r\zeta\right)\frac{d}{dz} + \frac{(-2r\zeta\sigma + r\eta - 2r\zeta + \sigma + \tau)}{z} + \frac{(-\frac{rm\omega}{\gamma^2} + r\zeta - \sigma - \tau)}{z-1}\right\}G(z) = 0\,, \tag{46}$$

which is the confluent Heun differential equation.[60,61] Let us denote the confluent Heun function by H_C, then the solutions can be written as

$$G(z) = C_1 H_C(a,b,c,d,e,z) + C_2 \exp(b) H_C(a,-b,c,d,e,z) \tag{47}$$

with

$$\begin{aligned}
a &= -2\left(\frac{\gamma(E+m)}{\varepsilon} + 1\right)\sqrt{\frac{(m^2\omega^2 - \varepsilon^2)}{\gamma^4}}\,,\\
b &= \frac{2}{\gamma}\sqrt{m^2 + \frac{m^2\omega^2}{\gamma^2} - \left(E + \frac{\varepsilon}{\gamma}\right)^2}\,, \quad c = -2\,,\\
d &= \left(\frac{\gamma(E+m)}{\varepsilon} + 1\right)\left(\frac{2(m^2\omega^2 - \varepsilon^2)}{\gamma^4} - \frac{2\varepsilon E}{\gamma^3}\right),\\
e &= -\left(\frac{\gamma(E+m)}{\varepsilon} + 1\right)\left(\frac{2(m^2\omega^2 - \varepsilon^2)}{\gamma^4} + \frac{m\omega}{\gamma^2} - \frac{2\varepsilon E}{\gamma^3}\right) + \frac{m\omega}{\gamma^2} + 1\,.
\end{aligned} \tag{48}$$

Then, the final expression for $\phi_1(x)$ is

$$\phi_1(x) = (1+\gamma x)^\sigma \exp(-\zeta(1+\gamma x))\left[C_1 H_C\left(a,b,c,d,e,\frac{(1+\gamma x)}{r}\right) + C_2 \exp(-b) H_C\left(a,-b,c,d,e,\frac{(1+\gamma x)}{r}\right)\right]. \tag{49}$$

Using the relation (33) and the expression of $\phi_1(x)$, we also find

$$\phi_2(x) = \frac{-i}{E+m-\varepsilon x}\left((1+\lambda x)\frac{d}{dx} + \frac{\lambda}{2} + m\omega x\right)\phi_1(x). \tag{50}$$

In order to have a polynomial solution for the confluent Heun equation, we need to cut the series which are given by the recurrence relation. For a polynomial solution of degree N, we impose,[60]

$$\frac{d}{a} + \frac{b+c}{2} + N + 1 = 0. \tag{51}$$

Using the condition (51) and replacing the parameters a, b and c by their expressions (48), we finally get the following energy spectrum

$$E^\pm = -\varepsilon\gamma\frac{\Omega N}{m\omega} \pm \Omega\sqrt{m^2 + 2m\omega\Omega N - \gamma^2\Omega^2 N^2} \quad \text{with} \quad \Omega = \frac{\sqrt{(m^2\omega^2 - \varepsilon^2)}}{m\omega}. \tag{52}$$

In this case, one notes practically the same remarks of the Klein–Gordon oscillator case. The expression of the energy spectrum contains all corrections of all orders of $(\varepsilon\gamma)^2$. This is related with the exact contribution to the Stark effect in this deformation framework and it varies with the power of N^2, which explains the confinement phenomenon. For large values of N, the square of the energy spectrum $(E)^2$ becomes negative and, in order to ensure positivity of the square of the energy, one must impose an upper bound on the allowed values of N.

Expanding the energy spectrum up to first order in γ^2, we obtain

$$E^\pm = \pm\Omega\sqrt{m^2 + 2m\omega\Omega N} - \varepsilon\gamma\frac{\Omega N}{m\omega} \mp \frac{\gamma^2\Omega^3 N^2}{2\sqrt{m^2 + 2m\omega\Omega N}}. \tag{53}$$

The first term in (53) is the energy spectrum of the usual Dirac oscillator subject to a uniform electric field. The second and the third terms represent the quantum fluctuations due to the new type of EUP.

We can also present our results for the Dirac oscillator graphically for some numerical values of the physical parameters. We will take $m = 1$ and $\omega = 10$ in our analysis. We will plot the curves only for E^+ as the curves for E^- do not show different physical behavior. One can easily see that the energy behavior is the same as in the Klein–Gordon oscillator case.

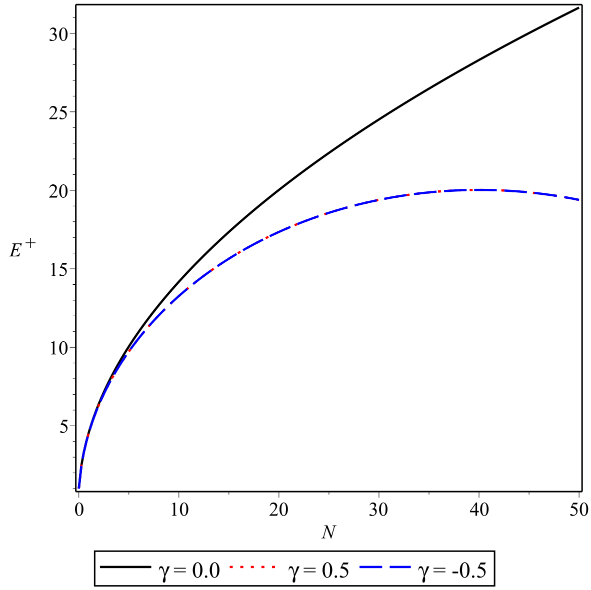

Fig. 5. (E^+ versus N) for $\varepsilon = 0$ (Dirac oscillator).

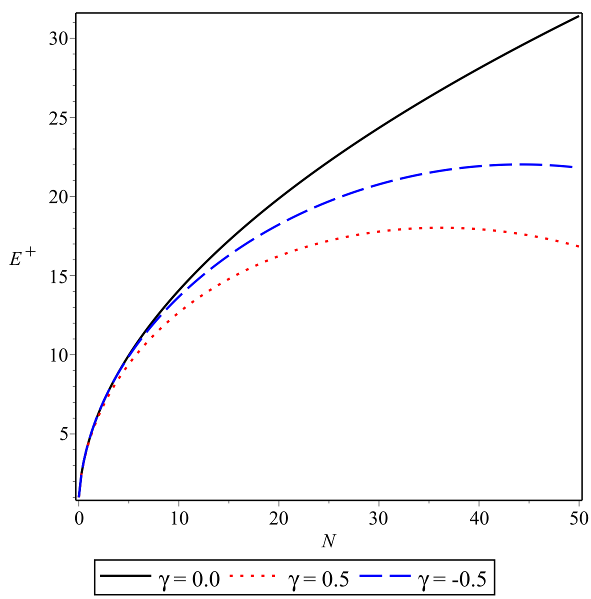

Fig. 6. (E^+ versus N) for $\varepsilon = 1$ (Dirac oscillator).

In Fig. 5, we plot the energy levels as a function of quantum number N for various values of γ and for $\varepsilon = 0$. We see that the values for nonzero γ coincide. If we take a fixed but nonzero ε as in Fig. 6, we find that the energy behavior is

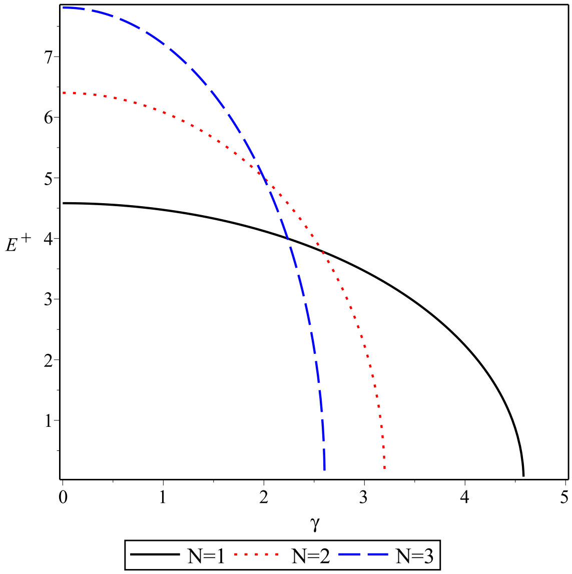

Fig. 7. (E^+ versus γ) for $\varepsilon = 0$ (Dirac oscillator).

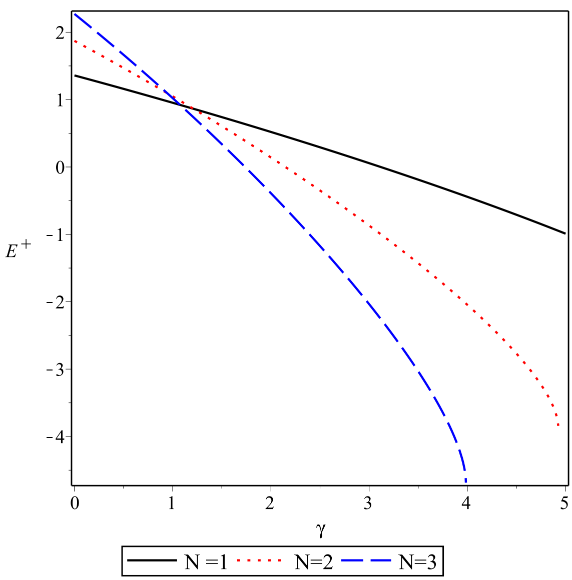

Fig. 8. (E^+ versus γ) for $\varepsilon = 9$ (Dirac oscillator).

different. The nonzero electric field yields a physical effect on the system. Figures 7 and 8 show the behavior of the energy for varying γ and for a fixed ε (we used $\varepsilon = 0$ and $\varepsilon = 9$, respectively). Here, we see the effect of γ on the energy behavior for some fixed N values.

3.1. *Some special cases*

We will consider some special cases for vanishing γ and ε.

For $\gamma = 0$, namely, in the absence of deformation, we replace $\gamma = 0$ in (52):

$$E^{\pm} = \pm\Omega\sqrt{m^2 + 2m\omega\Omega N}\,. \tag{54}$$

The case for $\varepsilon = 0$, namely, in the absence of an electric field, implies $\Omega = 1$, and the expression of the energy spectrum (52) becomes

$$E^{\pm} = \pm\sqrt{m^2 + 2m\omega N - \gamma^2 N^2}\,. \tag{55}$$

In the case where $\gamma = \varepsilon = 0$, we have the pure Dirac oscillator case. This limit yields

$$E = \pm\sqrt{m^2 + 2Nm\omega}\,, \tag{56}$$

which is in agreement with the result of the ordinary case.

4. Conclusion

In this paper, we studied the exact solutions of one-dimensional Klein–Gordon and Dirac oscillators subject to a uniform electric field in the context of the new type of the EUP using the displacement operator method.

The energy eigenvalues and eigenfunctions are determined for both cases. In the Klein–Gordon oscillator case, the wave functions are expressed in terms of the associated Laguerre polynomials and in the Dirac oscillator case, the wave functions are obtained in terms of the confluent Heun functions. In the latter case, the energy eigenvalues are obtained by the polynomial reduction of the confluent Heun functions.

The analytical expression of the energy spectrum contains corrections of all orders of $(\varepsilon\gamma)^2$. This is related to the exact contribution to the Stark effect in this deformation framework and it varies with the power of n^2, which explains the confinement phenomenon. For large values of n, the square of the energy spectrum $(E)^2$ becomes negative and, in order to ensure positivity of the square of the energy, one must impose an upper bound on the allowed values of n. The energy eigenvalues are plotted as a function of n for various numerical values of the parameter γ in order to show our result graphically.

The limiting cases are also studied using the special values of the physical parameters for both the Klein–Gordon and Dirac oscillator. It is remarkable that the results obtained in this context of the displacement operator can be interpreted as the case of systems with variable masses depending on the position. This study really needs more details, which will form the goal of a future project.

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Spinless Relativistic Particle in the Presence of Minimal Uncertainty in the Momentum

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Abstract In this letter, we present the exact solution of the three-dimensional Klein–Gordon oscillator on the (anti)-de Sitter spaces, the energy spectrum and the associated wave functions are extracted and the wave functions are expressed according to the Jacobi polynomial. On the other hand, we have investigated the three-dimensional the Klein–Gordon equation with a Coulomb plus scalar potential, we use the perturbation theory to calculate corrections to the spectrum in this framework of the extended uncertainty principle.

1 Introduction

Quantum field theory is the unfinished coronation of quantum mechanics and the laws of relativity. In spite of the exploit of its experimental predictions, it remains full of divergences which one could not eliminate except by methods of regularization mathematics and physical renormalization. Moreover, with the developments of new theories such as string theory [1], black hole physics [2], and quantum gravity [3], it turns out that a fundamental minimal length is required in a more unifying approach to existing physical interactions. The minimal length approach is one that best approximates the explicit computations and generalization of this quantum field theory to include the gravitational field. In addition, the existence of a minimal length leads to generalized uncertainty principle(GUP) [4–6] and modifies the canonical Heisenberg algebra to a non-canonical one. Mignemi in these research works [7,8] showed that it can be derived from the definition of quantum mechanics on a de Sitter background with a suitably chosen parametrization, that is, the Heisenberg uncertainty principle should be modified in a (anti)- de Sitter background by introducing corrections proportional to the cosmological constant $\wedge = \frac{3}{R^2}$, where $R^2 \prec 0$ for de Sitter space-time, and $R^2 \succ 0$ for anti-de Sitter space-time [9]. This modification was called extended uncertainty principle (EUP), it can be achieved by modifying the usual canonical commutation relations.

Over past decades, the implications of this (EUP) hypothesis have developed significantly and many works are examined for quantum mechanics and classical on the background (anti) -de Sitter [9–17].

In this analysis, first we are interested to study two fundamental problems of quantum mechanics in the context of (anti)-de Sitter spaces :

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– To establish the exact solutions of the $(1+3)$-dimensional Klein–Gordon oscillator.
– To determine the corrections to the spectrum of the Klein–Gordon equation for the coulomb plus scalar potentials using the perturbation theory. This gives rise to the appearance of a minimal uncertainty in momentum. On the other, we also study the effect of the deformation and the changes made to relativistic system in the framework of the extended uncertainty principle.

The outline of this letter is organized as follows: In Sect. 1, we give brief reminder of de Sitter and anti de Sitter space. In Sect. 2, we study the $(1+3)$-dimensional Klein–Gordon oscillator. In Sect. 3, we present the perturbative calculation of the spectrum of 3-dimensional hydrogen atom .

2 Review of the Deformed Quantum Mechanics Relation: de Sitter and Anti-de Sitter Spaces

The extended uncertainty principle (EUP) can be obtained from the definition of quantum mechanics on (anti) de sitter space-time. It is well known that (anti)-de Sitter space-time can be realized as a hyperboloid of equation $\eta_{ab}\zeta^a\zeta^b = \pm R^2$ embedded in five-dimensional Minkowski space with coordinates $\zeta^a (a = 0, 1, 2, 3, 4)$ and metric $\eta_{ab} = diag\,(1, -1, -1, -1, \pm 1)$, when $R \to \infty$ the de Sitter (dS) invariant special relativity (SR) will be reduced to ordinary special relativity [18]

$$ds^2 = \eta_{ab} d\zeta^a d\zeta^b = B_{\mu\nu}(x)\, dx^\mu dx^\nu;\;\; \mu = \nu = 0; 1; 2; 3, \tag{1}$$

where the parametrization of the hyperboloid is given by projective (Beltrami) coordinates [19,20],

$$x_\mu = \frac{\zeta_\mu}{\zeta_4} \tag{2}$$

and

$$B^{\mu\nu}(x) = \left(1 - \frac{\eta_{\sigma\tau} x^\sigma x^\tau}{R^2}\right)\left(\eta^{\mu\nu} - \frac{x^\mu x^\nu}{R^2}\right), \tag{3}$$

is Beltrami metric. Note that, the Beltrami coordinate system, is similar to the Minkowski one in a flat space-time, and the Beltrami de sitter ($\mathcal{B}dS$) space-time is the dS space-time with Beltrami metric. The generators of de Sitter in Beltrami coordinates and the momentum operators satisfy the following commutation relations [7,8,21–23]

$$\left[J_{\mu\nu}, J_{\sigma\rho}\right] = i\left(\eta_{\nu\rho} J_{\mu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma}\right), \tag{4}$$

$$\left[J_{\mu\nu}, p_\rho\right] = i\left(\eta_{\mu\rho} p_\nu - \eta_{\nu\rho} p_\mu\right);\quad \left[p_\mu, p_\nu\right] = \frac{i J_{\mu\nu}}{R^2}, \tag{5}$$

and

$$\left[x_\mu, p_\nu\right] = i\left(\eta_{\mu\nu} + \frac{x_\mu x_\nu}{R^2}\right);\quad \left[x_\mu, x_\nu\right] = 0. \tag{6}$$

where $\mu\ \nu = 0,1,2,3$ and $J_{\mu\nu}$ are the generators of Lorentz transformations given by $J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$

In the theory of SR on (A)dS space-time there are two universal parameters: the speed of light c and the cosmological constant $\wedge$ [18].

The non-relativistic modified commutation relations leading to the extended commutation relations, is given by [12]

$$\begin{cases} \left[X_j, P_k\right] = i\hbar\left(\delta_{jk} + \alpha X_j X_k\right), \\ \left[X_j, X_k\right] = 0, \\ \left[P_j, P_k\right] = i\hbar\alpha L_{jk}, \end{cases} \tag{7}$$

where $j, k = 1, 2, 3$, $L_{jk} = X_j P_k - X_k P_j$, and α being the constant deformation parameter, where α is a positive parameter proportional to the cosmological constant or inversely proportional to the square of the anti-de Sitter radius ($\alpha = H^2 : H^2$ is the Hubble rate) [24], and in the limit $\alpha \longrightarrow 0$, we recover the canonical commutation relations from standard quantum mechanics.

As the case of ordinary quantum mechanics, the commutation relation (7) lead to the following extended uncertainty principle (EUP)

$$(\Delta X_i)(\Delta P_i) \geq \frac{\hbar}{2}\left(1+\alpha(\Delta X_i)^2\right), \tag{8}$$

which implies the appearance of a nonzero minimal uncertainty in momentum. The minimization of (8) with respect to ΔX_i gives

$$(\Delta P_i)_{\min} = \hbar\sqrt{\alpha}, \quad \forall \quad k. \tag{9}$$

The most representation of the position and momentum operators obeying relation (7) is given by

$$X_i = x_i; \quad P_i = \frac{\hbar}{i}\left(\delta_{ij}+\alpha x_i x_j\right)\frac{\partial}{\partial x_i}, \tag{10}$$

where the operators x_i and p_j satisfy the canonical commutation relation $\left[x_i, p_j\right] = i\hbar\delta_{ij}$. Using the symmetrically condition of the operators of position and momentum, the modified scalar product can be written as

$$\langle\phi\,|\psi\rangle = \int \frac{d^3\mathbf{r}}{\left(1+\alpha r^2\right)^2}\phi^{\times}(\mathbf{r})\,\psi(\mathbf{r})\,; \text{ where } r = \sum_{i=1}^{3} x_i^2. \tag{11}$$

Now, the extended uncertainty principle for the de Sitter space, which can be constructed by replacing $\alpha \longrightarrow -\alpha$, in this case and contrary to the previous case, we will have,

$$(\Delta X_i)(\Delta P_i) \geq \frac{\hbar}{2}\left(1-\alpha(\Delta X_i)^2\right). \tag{12}$$

let's notice this relation does not give the minimal uncertainty in momentum, we get

$$-\frac{(\Delta P_i)}{\alpha\hbar} - \frac{1}{\alpha}\sqrt{\alpha+\frac{(\Delta P_i)^2}{\hbar^2}} \leq (\Delta X_i) \leq -\frac{(\Delta P_i)}{\alpha\hbar} + \frac{1}{\alpha}\sqrt{\alpha+\frac{(\Delta P_i)^2}{\hbar^2}}. \tag{13}$$

and in the limit $(\Delta P_i) \to 0$ the space become finite $-\frac{1}{\sqrt{\alpha}} \leq (\Delta X_i) \leq \frac{1}{\sqrt{\alpha}}$.

A representation of X_i and P_i that satisfies for the de Sitter space, may be taken as

$$X_i = x_i; \; P_i = \frac{\hbar}{i}\left(\delta_{ij}-\alpha x_i x_j\right)\frac{\partial}{\partial x_j}. \tag{14}$$

In the following section, we examine the Klein Gordon oscillator and The Klein–Gordon equation with a Coulomb plus scalar potential in anti-de Sitter space. we put ($\hbar = c = 1$)

3 (1+3)-Dimensional Klein Gordon Oscillator in AdS Space

In this section, we are interested in solving the (1 + 3)-dimensional Klein Gordon oscillator, in position space with deformed commutation relations. In this case, the stationary equation describing the Klein Gordon oscillator in (1 + 3)-dimension is given by

$$\left[\left(E^2-m^2\right)-(\mathbf{P}+im\omega\mathbf{r})(\mathbf{P}-im\omega\mathbf{r})\right]\Phi(\mathbf{r}) = 0, \tag{15}$$

where m is the rest mass, and ω is the classical frequency of the oscillator.

Applying the definition of the position and momentum operators reported in Sect. (2), the momentum squared operator can be expressed as

$$P^2 = -\left[\left(1+\alpha r^2\right)\frac{\partial}{\partial r}\right]^2 - \frac{2}{r}\left(1+\alpha r^2\right)\frac{\partial}{\partial r} + \frac{L^2}{r^2} \tag{16}$$

and the Klein Gordon oscillator equation (15) can be rewritten as the following differential equation:

$$\left(m^2-E^2\right)\Phi=\left\{\left[\left(1+\alpha r^2\right)\frac{\partial}{\partial r}\right]^2+\frac{2}{r}\left(1+\alpha r^2\right)\frac{\partial}{\partial r}-\frac{\hat{L}^2}{r^2}-m^2\omega^2r^2+m\omega\left(3+\alpha r^2\right)\right\}\Phi\left(\mathbf{r}\right). \quad (17)$$

Thus, it's appropriate to split the energy eigenfunction Φ into a radial part and an angular part as:

$$\Phi\left(\mathbf{r}\right)=R_{n,\ell}\left(r\right)Y_{\ell,m}\left(\theta,\varphi\right), \quad (18)$$

where $Y_{n,\ell}$ are the eigenfunction of the angular part.

$$\hat{L}^2Y_{\ell,m}\left(\theta,\varphi\right)=\ell\left(\ell+1\right)Y_{\ell,m}\left(\theta,\varphi\right) \quad (19)$$

This allows us to rewrite Eq. (17) as

$$\left[\left[\left(1+\alpha r^2\right)\frac{d}{dr}\right]^2+\frac{2}{r}\left(1+\alpha r^2\right)\frac{d}{dr}-\frac{\ell\left(\ell+1\right)}{r^2}-m^2\omega^2r^2+m\omega\alpha r^2+E^2-m^2+3m\omega\right]R_{n,\ell}\left(r\right)=0. \quad (20)$$

To solve this equation, we begin by making the following change of variable

$$\sqrt{\alpha}\rho=\tan^{-1}\sqrt{\alpha}r, \quad (21)$$

which maps the interval $r\in]0,\infty[$ to $\rho\in]0,\frac{\pi}{2\sqrt{\alpha}}[$ and brings Eq. (20) to the following form

$$\left[\frac{d^2}{d\rho^2}+\frac{2\sqrt{\alpha}}{\tan\left(\sqrt{\alpha}\rho\right)}\frac{d}{d\rho}-\frac{\alpha\ell\left(\ell+1\right)}{\tan^2\left(\sqrt{\alpha}\rho\right)}-m\omega\left(\frac{m\omega}{\alpha}-1\right)\tan^2\left(\sqrt{\alpha}\rho\right)+E^2-m^2+3m\omega\right]R_{n,\ell}\left(\rho\right)=0. \quad (22)$$

To eliminate the first derivative, we introduce the following ansatz

$$R_{n,\ell}\left(\rho\right)=e^{-\sqrt{\alpha}\int^{\rho}\frac{d\zeta}{\tan\left(\sqrt{\alpha}\zeta\right)}}g_{n,\ell}\left(\rho\right), \quad (23)$$

after some manipulation, we obtain

$$\left[\frac{d^2}{d\rho^2}-\frac{\alpha\ell\left(\ell+1\right)}{\tan^2\left(\sqrt{\alpha}\rho\right)}-m\omega\left(\frac{m\omega}{\alpha}-1\right)\tan^2\left(\sqrt{\alpha}\rho\right)+E^2-m^2+3m\omega+\alpha\right]g_{n,\ell}\left(\rho\right)=0. \quad (24)$$

Introducing now the following change of function

$$g_{n,\ell}\left(\rho\right)=\sin^{\ell+1}\left(\sqrt{\alpha}\rho\right)\cos^{\sigma}\left(\sqrt{\alpha}\rho\right)F_{n,\ell}\left(\rho\right), \quad (25)$$

where σ is a constant to be determined letter. By means of the substitution given in Eq. (25), the last differential equation (24) take the following form:

$$\left[\begin{array}{c}\frac{d^2}{d\rho^2}+2\sqrt{\alpha}\left(\frac{(\ell+1)}{\tan\left(\sqrt{\alpha}\rho\right)}-\sigma\tan\left(\sqrt{\alpha}\rho\right)\right)\frac{d}{d\rho}-\alpha\sigma\left(2\ell+3\right)\\+\alpha\left[\sigma\left(\sigma-1\right)-\frac{m\omega}{\alpha}\left(\frac{m\omega}{\alpha}-1\right)\right]\tan^2\left(\sqrt{\alpha}\rho\right)+E^2-m^2+3m\omega-\ell\alpha\end{array}\right]F_{n,\ell}\left(\rho\right)=0. \quad (26)$$

To eliminate the term $\tan^2\left(\sqrt{\alpha}\rho\right)$ by demanding

$$\sigma\left(\sigma-1\right)-\frac{m\omega}{\alpha}\left(\frac{m\omega}{\alpha}-1\right)=0, \quad (27)$$

then it leads to the following expression of σ

$$\sigma_+=\frac{m\omega}{\alpha},\sigma_-=1-\frac{m\omega}{\alpha}. \quad (28)$$

Among these two solutions, the physically acceptable one is only σ_+, the second solution leads to a non physically acceptable wave function. Then Eq. (26) simplifies to

$$\left[\frac{d^2}{d\rho^2}+2\sqrt{\alpha}\left(\frac{\ell+1}{\tan\left(\sqrt{\alpha}\rho\right)}-\frac{m\omega}{\alpha}\tan\left(\sqrt{\alpha}\rho\right)\right)\frac{d}{d\rho}-2\ell m\omega+E^2-m^2-\alpha\ell\right]F_{n,\ell}\left(\rho\right)=0. \tag{29}$$

At this stage, we introduce another change of variable defined by

$$\eta=2\sin^2\left(\sqrt{\alpha}\rho\right)-1.\text{ with }-1\preceq\eta\preceq 1 \tag{30}$$

the Eq. (29) reduces to

$$\left[\left(1-\eta^2\right)\frac{d^2}{d\eta^2}+\left(\ell-\frac{m\omega}{\alpha}+1-\left(\ell+\frac{m\omega}{\alpha}+2\right)\eta\right)\frac{d}{d\eta}+\frac{E^2-m^2-\alpha\ell-2\ell m\omega}{4\alpha}\right]F_{n,\ell}\left(\eta\right)=0. \tag{31}$$

which is exactly the Jacobi polynomials differential equation $P_n^{(a,b)}(\eta)$ whose parameters a and b are given by imposing the following constraint

$$\frac{E^2-m^2-\alpha\ell-2\ell m\omega}{4\alpha}=n\left(n+a+b+1\right), \tag{32}$$

$$a=\frac{m\omega}{\alpha}-\frac{1}{2};\quad b=\ell+\frac{1}{2}. \tag{33}$$

where n is non-negative integer and the solution can be written in terms of Jacobi polynomials as

$$F_{n,\ell}\left(\eta\right)=P_n^{\left(\frac{m\omega}{\alpha}-\frac{1}{2},\ell+\frac{1}{2}\right)}\left(\eta\right). \tag{34}$$

Using the the former variable r, we will have the following final form of the wave function Φ :

$$\Phi_{n,\ell}\left(\mathbf{r}\right)=\mathcal{C}\frac{r^\ell}{\left(1+\alpha r^2\right)^{\frac{m\omega}{2\alpha}+\frac{\ell}{2}}}P_n^{\left(\frac{m\omega}{\alpha}-\frac{1}{2},\ell+\frac{1}{2}\right)}\left(\frac{\alpha r^2-1}{1+\alpha r^2}\right)Y_{\ell,m}\left(\theta,\varphi\right), \tag{35}$$

where $\mathcal{C}$ is the normalization constant.

To determine the expressions of the energy spectrum of Klein Gordon oscillator , using the condition (32) and replacing the parameters a, and b by their expressions (33), we finally get the following result

$$E_{n,l}^{\pm AdS}=\pm\sqrt{m^2+2m\omega\left(2n+\ell\right)+\alpha\left[4n\left(n+l+1\right)+l\right]}, \tag{36}$$

where $\pm$ denotes the positive (negative) energy solutions associated respectively with the particle and the antiparticle for relativistic quantum systems.

Notice that the energy levels depend on the quantum number n and n^2 and for large n it is asymptotic to

$$E_n^{\pm AdS}\rightarrow\pm 2\sqrt{\alpha}n, \tag{37}$$

This effect is due to the modification of the Heisenberg algebra. As a result, we remark that for a fixed value of n, the energy $E_{n,l}^{+AdS}$ increases monotonically with the increase of the EUP parameter α. Expanding the expression of the energy levels to first order in α, we obtain

$$E_{n,l}^{\pm AdS}=\pm\sqrt{m^2+2m\omega\left(2n+\ell\right)}\left(1+\frac{\alpha}{2}\frac{\left(4n\left(n+l+1\right)+l\right)}{\left(m^2+2m\omega\left(2n+\ell\right)\right)}\right) \tag{38}$$

The first term is the energy spectrum of the ordinary 3d Klein–Gordon oscillator, while the second term is the corrections brought about by the existence of nonzero minimal uncertainty in momentum, and when we study the limit $\alpha\rightarrow 0$, we obtain

$$E_{n,l}^{\pm AdS}=\pm\sqrt{m^2+2m\omega\left(2n+\ell\right)} \tag{39}$$

which is the same result in ordinary case. Before finishing this section, let us see the influence of the EUP in dS on the energy eigenvalues ($\alpha \to -\alpha$) and by the same steps and same techniques, we arrive

$$E_{N,j}^{\pm dS} = \pm\sqrt{m^2 + 2m\omega\,(2n+\ell) - \alpha\,[4n\,(n+l+1)+l]}, \tag{40}$$

In this case, for large values of n, the square of the energy spectrum $\left(E_{n,\mu}^{dS}\right)^2$ becomes negative. In order to ensure positivity of the the square of the energy, one must impose an upper bound on the allowed values n and l.

4 The Klein–Gordon Equation with a Coulomb Plus Scalar Potential in AdS Space

The hydrogen atom is a fundamental problem of quantum mechanics; it is of considerable importance in atomic and molecular physics. it allows to understand the spectra of hydrogenoids and to explain the structure of the energy levels and the spectra of the atoms in the case of models with independent electrons or approach of an average field. Furthermore, the hydrogen atom has grown enormously;especially in the context of deformed algebras and several papers have been studied. In non relativistic case, the spectrum and eigenfunctions in the momentum representation for 1D Coulomb-like potential with deformed Heisenberg algebra are found exactly in [25,26], for higher dimensions, the problem becomes complicated, only perturbative solutions have been found [27–30]. On the contrary, in the case of relativistic quantum mechanics, no study is presented, accordingly, our attempt through this letter will be addressed the problem in question for the case of the Klein Gordon equation in the framework of anti-de Sitter spaces.

To study the eigenvalue problem for hydrogen atom in 3-dimensional case we start considering a standard Hamiltonian:

$$\left\{P^2 + (M + V_s(r))^2 - (E + V_v(r))^2\right\}\psi(\mathbf{r}) = 0 \tag{41}$$

where M and E denote the mass and the energy of the particle, respectively and $r = \sqrt{\sum_{j=1}^{3} X_j^2}$ and $P = \sqrt{\sum_{j=1}^{3} P_j^2}$ satisfy deformed commutation relation (7). The Coulomb potential and the scalar potential are taken as

$$V_s(r) = -\frac{V_s}{r} \quad V_v(r) = -\frac{V_v}{r} \tag{42}$$

The scalar potential is added to the mass term in the Klein Gordon equation and may be understood as an effective position-dependent mass, which is of considerable significance in various areas of physics, citing for instance quantum well and quantum dots [31–33], in the description of electronic properties and band structure of semiconductor heterostructures [34,35], …etc.

Now, we apply the definition for X_j and P_j (10) reported in Sect. 2, the momentum squared operator (16) can be expressed

$$P^2 = -\left(1+\alpha r^2\right)^2 \frac{\partial^2}{\partial r^2} - \left(1+\alpha r^2\right) 2\alpha r \frac{\partial}{\partial r} - \frac{2}{r}\left(1+\alpha r^2\right)\frac{\partial}{\partial r} + \frac{L^2}{r^2}, \tag{43}$$

if we expand (43) at the first order in α, we have

$$P^2 = -\left(1+2\alpha r^2\right)\frac{\partial^2}{\partial r^2} - \frac{2}{r}\left(1+2\alpha r^2\right)\frac{\partial}{\partial r} + \frac{L^2}{r^2} + \mathcal{O}\left(\alpha^2\right), \tag{44}$$

therefore, the Klein Gordon equation (41) can be written as follows

$$\left\{-\left(1+2\alpha r^2\right)\frac{\partial^2}{\partial r^2} - \frac{2}{r}\left(1+2\alpha r^2\right)\frac{\partial}{\partial r} + \frac{L^2}{r^2} + \left(M - \frac{V_s}{r}\right)^2 - \left(E + \frac{V_v}{r}\right)^2\right\}\psi(\mathbf{r}) = 0, \tag{45}$$

or as follows ;

$$\left\{-\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) - 2\alpha r^2\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) + \frac{L^2}{r^2} + \left(M - \frac{V_s}{r}\right)^2 - \left(E + \frac{V_v}{r}\right)^2\right\}\psi(\mathbf{r}) = 0, \tag{46}$$

and using this replacement

$$\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}=\frac{1}{r}\frac{\partial^2}{\partial r^2}r, \tag{47}$$

the Eq. (46) becomes

$$\left\{-\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r\right)-2\alpha r^2\left(\frac{1}{r}\frac{\partial^2}{\partial r^2}r\right)+\frac{L^2}{r^2}+\left(M-\frac{V_s}{r}\right)^2-\left(E+\frac{V_v}{r}\right)^2\right\}\psi(\mathbf{r})=0. \tag{48}$$

To solve this equation, using this separate form ;

$$\psi(\mathbf{r})=\frac{R^{\alpha}(r)}{r}Y_{l,m}(\theta,\varphi). \tag{49}$$

where $Y_{l,m}(\theta,\varphi)$ are spherical harmonics, eigenvectors of the orbital kinetic moment

$$L^2Y_{l,m}(\theta,\varphi)=\ell(\ell+1)Y_{l,m}(\theta,\varphi)=\left(k^2-\frac{1}{4}\right)Y_{l,m}(\theta,\varphi) \tag{50}$$

with $k=l+\frac{1}{2}$. Substitution (49) and (50) into Eq. (48), We obtain the radial equation of the Klein–Gordon equation in AdS space:

$$\left[-\frac{d^2}{dr^2}+\frac{k^2+V_s^2-V_v^2-\frac{1}{4}}{r^2}+\left(M^2-E^2\right)-\frac{2\left(MV_s+EV_v\right)}{r}-2\alpha r^2\left(\frac{d^2}{dr^2}\right)+\mathcal{O}\left(\alpha^2\right)\right]R^{\alpha}(r)=0, \tag{51}$$

which can be written as

$$\left[H_0+\alpha W+\mathcal{O}\left(\alpha^2\right)\right]R^{\alpha}(r)=0. \tag{52}$$

with H_0 represents the undisturbed Hamiltonian corresponds to the ordinary case $\alpha=0$ of the Klein–Gordon equation for Hydrogen atom with scalar potential given by

$$H_0=-\frac{d^2}{dr^2}+\frac{k^2+V_s^2-V_v^2-\frac{1}{4}}{r^2}+\left(M^2-E^2\right)-\frac{2\left(MV_s+EV_v\right)}{r} \tag{53}$$

and W is the disturbed Hamiltonian

$$W=-2r^2\frac{d^2}{dr^2}. \tag{54}$$

To simplify the shape of H_0 and W , introducing this notation

$$\beta=\frac{EV_v+MV_s}{\sqrt{M^2-E^2}},\ \nu=\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}\text{ and }a=\sqrt{M^2-E^2}, \tag{55}$$

we will then have

$$H_0=-\frac{d^2}{dr^2}+\frac{\nu(\nu+1)}{r^2}+a^2-\frac{2a\beta}{r} \tag{56}$$

and the new the expression form for W,

$$W=-2\nu(\nu+1)+2r^2\left(H_0-a^2\right)+4ra\beta. \tag{57}$$

We have used (53).

In order to study the influence of this deformation on the energy levels of the hydrogen atom we will consider the term αW as perturbation in ordinary quantum mechanics. Therefore, the perturbation theory can be used to calculate the correction to the energy levels of the hydrogen atom in the first-order in α and to avoid complex spectra, subsequently, we consider the case of the weak Coulomb potential such that $k^2+V_s^2-V_v^2>0$, otherwise the solution becomes oscillatory .

Now, for $\alpha=0$, the exact solution of the ordinary Klein Gordon equation for Hydrogen atom can be found in [36,37]. The eigenvalues and the corresponding normalized eigenfunctions expressed according to Laguerre's polynomial are given by

$$R^0_{n'\nu}(r)=N_{n'l}\frac{n'!\Gamma(2\nu+2)}{\Gamma(2\nu+2+n')}(2ar)^{\nu+1}e^{-ar}L^{2\nu+1}_{n'}(2ar) \tag{58}$$

where $N_{n'l}$ is normalization constant determined by this condition

$$\int R^{0*}_{n'\nu}(r)R^0_{n'\nu}(r)dr=1. \tag{59}$$

By using the recursion relation for Laguerre polynomials [38]

$$xL^{2\nu+1}_n=2(n+\nu+1)L^{2\nu+1}_n-(n+1)L^{2\nu+1}_{n+1}-(n+2\nu+1)L^{2\nu+1}_{n-1}. \tag{60}$$

and

$$d_n^2=\int x^\alpha e^{-x}L^\alpha_n(x)\,L^\alpha_n(x)\,dx=\frac{\Gamma(n+\alpha+1)}{n!} \tag{61}$$

where d_n^2 is the square of the norm of $L^\alpha_n(x)$, the normalized radial functions are

$$R^0_{n'\nu}(r)=\sqrt{\frac{an'!}{(\nu+n'+1)\,\Gamma(2\nu+2+n')}}(2ar)^{\nu+1}e^{-ar}L^{2\nu+1}_{n'}(2ar) \tag{62}$$

and the corresponding energy spectrum, eigenvalues of the radial part of the Klein–Gordon equation with a Coulomb potential and scalar potential is deducted by this condition $\beta-\nu-1=n'$.

$$E^{\alpha=0\pm}_{n,\ell}=M\left\{-\frac{V_sV_v}{V_v^2+\beta^2}\pm\left[\left(\frac{V_sV_v}{V_v^2+\beta^2}\right)^2-\frac{V_s^2-\beta^2}{V_v^2+\beta^2}\right]^{\frac{1}{2}}\right\}, \tag{63}$$

or

$$E^{\alpha=0\pm}_{n,\ell}=M\left\{-\frac{V_sV_v}{V_v^2+(\nu+n-l)^2}\pm\left[\left(\frac{V_sV_v}{V_v^2+(\nu+n-l)^2}\right)^2-\frac{V_s^2-(\nu+n-l)^2}{V_v^2+(\nu+n-l)^2}\right]^{\frac{1}{2}}\right\}, \tag{64}$$

where we have introduced the principal quantum number : $n=n'+l+1$.

Now, to determine the correction of the energy levels associated with the disturbed Hamiltonian W (57) due to the anti-de sitter space-time, we use the first-order perturbation theory in the deformation parameter α,

$$\begin{aligned}\alpha E^{(1)}_n&=\alpha\int R^{0*}_{nl}(r)\,(W)\,R^0_{nl}(r)dr\\&=\alpha\left[-2\nu\,(\nu+1)<r^{(0)}>+4a\beta<r^{(1)}>-2a^2<r^{(2)}>\right]\end{aligned} \tag{65}$$

where

$$<r^{(m)}>\int r^mR^{0*}_{nl}(r)R^0_{nl}(r)dr. \tag{66}$$

For the calculation of expectation of value of $<r^{(m)}>$, we take advantage of the properties (60) and (61) and a straightforward and long calculation leads to

$$\left\{\begin{array}{c}<r^{(0)}>=1\\<r^{(1)}>=\int rR^{0*}_{nl}(r)R^0_{nl}(r)dr=\frac{1}{2a(\nu+n-l)}\left[3(\nu+n-l)^2-\nu\,(\nu+1)\right]\\<r^{(2)}>=\int r^2R^{0*}_{nl}(r)R^0_{nl}(r)dr=\frac{1}{2a^2}\left[5(\nu+n-l)^2+1-3\nu\,(\nu+1)\right]\end{array}\right. \tag{67}$$

Then the first order of the perturbation theory takes this form

$$\begin{aligned}\alpha E_n^{(1)} &= \alpha\left[(\nu+n-l)^2-\nu(\nu+1)-1\right]\\ &= \alpha\left\{\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2\right.\\ &\quad\left.-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}\right)\left(\sqrt{k^2+V_s^2-V_v^2}+\frac{1}{2}\right)-1\right\}\end{aligned} \tag{68}$$

which represents the quantum fluctuations due to the extended uncertainty principle on (anti) -de sitter space-time, depending on the powers in n^2, explains the phenomenon of confinement and the expression of the hydrogen atom energy levels is modified as

$$\begin{aligned}{}^{AdS}E_{n,\ell}^{\alpha\pm} = M&\left\{-\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right.\\ &\pm\left[\left(\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right)^2\right.\\ &\left.\left.-\frac{V_s^2-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right]^{\frac{1}{2}}\right\},\\ &+\alpha\left\{\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2\right.\\ &\left.-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}\right)\left(\sqrt{k^2+V_s^2-V_v^2}+\frac{1}{2}\right)-1\right\}+\mathcal{O}\left(\alpha^2\right).\end{aligned} \tag{69}$$

In this last expression of the spectrum (69), we notice that the spectrum energy on anti-de Sitter is bigger than the energy in ordinary case. Before concluding this paragraph, we would like to see the influence of space dS on the eigenvalues of the system ($\alpha\to-\alpha$). By the same steps, the energy eigenvalues of the system will have the following form

$$\begin{aligned}{}^{ds}E_{n,\ell}^{\alpha\pm} = M&\left\{-\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right.\\ &\pm\left[\left(\frac{V_sV_v}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right)^2\right.\\ &\left.\left.-\frac{V_s^2-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}{V_v^2+\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2}\right]^{\frac{1}{2}}\right\},\\ &-\alpha\left\{\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}+n-l\right)^2\right.\\ &\left.-\left(\sqrt{k^2+V_s^2-V_v^2}-\frac{1}{2}\right)\left(\sqrt{k^2+V_s^2-V_v^2}+\frac{1}{2}\right)-1\right\}+\mathcal{O}\left(\alpha^2\right).\end{aligned} \tag{70}$$

In this case, we can see that the energy spectrum (70) on the de Sitter space is smaller than the energy in ordinary case.

5 Conclusion

In this contribution, we have investigated the three-dimensional Klein–Gordon oscillator and the Klein–Gordon equation with a Coulomb plus scalar potential in the context of quantum deformations for the (anti)- de Sitter algebras. For the 3-dimensionals Klein–Gordon oscillator, according to the symmetry of the system, we used the adequate radial representation and some change of variables, the problem has been converted to a differential equation of type Jacobi polynomials. The energy eigenvalues and their corresponding eigenfunctions are exactly and analytically obtained . For the case of the Klein–Gordon equation for hydrogen atom, the problem is complicated and in order to determine energy spectra, the perturbation theory has been applied to calculate the correction to the energy levels in the first-order in α. In both problem, we show that the spectrum energy on anti-de Sitter is bigger than the energy in ordinary case contrariwise in ds space , the energy spectrum is smaller than the energy in ordinary case.

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Abstract

Our thesis is essentially composed of two parts.

- In the first part, some problems are presented via the direct method of equations such:

In the context of new type of the extended uncertainty principle, using the displacement operator method, the exact solution of the Klein Gordon equation is given in the following cases: in a one dimensional box, with linear vector and scalar potentials and in mixed Coulomb-type vector and scalar potentials. The one-dimensional Klein-Gordon and Dirac oscillators subject to a uniform electric field.

In the context of the deformed Snyder-de Sitter model, the three-dimensional Klein-Gordon oscillator and the Klein-Gordon equation with a Coulomb plus scalar potential are traited.

- The second part is devoted especially to supersymmetric path integrals approach within the framework of the EUP to establish the function of Green for the oscillator of Dirac to (1 + 1) dimension. Following the global representation and Schwinger's proper time method, Green's causal function is obtained. By an adequate choice of the discretization of the measure and of the action, the appropriate quantum fluctuations are determined, and with the help of appropriate transformations, the propagator has converted to the case of the standard problem of the Poschl-Teller potential.

In all cases, the energy spectra and the corresponding wave functions are exactly and analytically determined and the obtained results agree with those of the literature . Also the limiting cases are considered.

Keywords: Deformed algebra, Path integral approach, Minimal length, Displacement Operator, The Translation Operator, Klein-Gorden and Dirac Oscillateurs.

Résumé

Notre thèse est composée essentiellement de deux parties.

- Dans la première partie, certains problèmes de la mécanique quantique realtiviste avec et sans spin sont présentés via la méthode directe des équations ;tels que:

Dans le contexte d'un nouveau type de principe d'incertitude étendue, en utilisant la méthode de l'opérateur de déplacement, la solution exacte et analytique de l'équation de Klein Gordon est donnée dans les cas suivants:

Particule confinée dans une boite, potentiels scalaire et vectoriel de types lineaires et Coulombiens. Les oscillateurs Klein-Gordon et Dirac soumis à un champ électrique uniforme.

Dans le contexte du modèle déformé de Snyder-de Sitter, l'oscillateur tridimensionnel de Klein-Gordon et l'équation de Klein-Gordon en présence du potentiel Coulombien sont traités.

- Alors la deuxième partie est consacrée spécialement à l'approche des intégrales de chemins supersymétriques dans le cadre de l'EUP pour établir la fonction de Green pour l'oscillateur de Dirac à (1+1) dimension. Suite à la représentation globale et la méthode du temps propre de Schwinger, la fonction causale de Green est obtenue. Par un choix adéquat de la discrétisation de la mesure et de l'action, les fluctuations quantiques appropriées sont déterminées, et à l'aide de transformations appropriées, le propagateur s'est converti au cas du problème standard du potentiel de Poschl-Teller.

Dans tous les cas, les spectres énergétiques et les fonctions d'onde associées sont exactement déterminés et concordent avec ceux de la littérature. Les cas particuliers sont aussi considérés.

Mots-clés : Algèbre déformée, Approche des intégrales de chemins supersymétriques, Longueur minimale, Opérateur de déplacement, Klein-Gorden et Dirac Oscillateurs.

الملخص:

تتكون أطروحتنا بشكل أساسي من جزأين.

- في الجزء الأول ، يتم عرض بعض مسائل ميكانيك الكم النسبي مع وبدون عزم اللف بواسطة طريقة المعادلات المباشرة ؛ مثل:

في سياق نوع جديد من مبدأ الشك الممتد و باستعمال طريقة مؤثر الانسحاب ، تم تقديم الحل الدقيق والتحليلي لمعادلة كلاين جوردون في الحالات التالية:

الجسيمات محصورة في صندوق ، الكمون العددي والشعاعي من النوعين الخطي والكولوم. هزازي كلاين جوردون وديراك خاضعة لمجال كهربائي ثابت.

و في سياق نموذج سنايدر دو سير المشوه ، تمت معالجة هزاز كلاين جوردون ثلاثي الأبعاد ومعادلة كلاين جوردون في وجود كمون من النوع كولوم.

- كما تم تخصيص الجزء الثاني بشكل خاص لمقاربة تكاملات المسارات فائقة التماثلة في إطار مبدأ الشك الممتد لتشكيل دالة جرين لهزازديراك على البعد (1 + 1). باتباع التمثيل الشامل وتقنية الزمن الذاتي لشوينغر ، تم الحصول على دالة جرين السببية. من خلال الاختيار الأنسب لتقدير القياس والفعل ، تم إيجاد التصحيحات الكمية المناسبة ، وبواسطة التحولات الملائمة ، تحول الناشر إلى حالة كمون بوشل تيلار العادية .

في كل الحالات تم تحديد اطياف الطاقة و الدوال الموجية المرفقة بشكل دقيق و التي تتفق مع النتائج الموجودة في المراجع . و اخيرا تم النظر في الحالات الخاصة.

الكلمات المفتاحية: الجبر المشوه ، نهج تكاملات المسارات فائقة التناظر ، الحد الأدنى للطول ، مؤثر الانسحاب ، هزازات كلاين-جوردن وديراك.